The Van der Pol Oscillator

(Ref: Guckenheimer and Holmes, 1983: *Nonlinear Oscillations, Dynamical Systmes and Bifurcations of Vector Fields*. Springer Verlag, pp. 66-82.)

As our next example we consider the Van der Pol oscillator, an oscillator with nonlinear damping which is negative for small amplitude oscillations and positive for large amplitude oscillations. Originally, this was a model for an electrical circuit with a triode whose resistance goes from negative to positive with increasing current. It is also a simple model for the effect of aerodynamic flutter on a wing. Similar types of equations arise in biochemical systems such the insulin regulating cells in the pancreas. The Van der Pol equation is

$$\ddot{y} + \alpha(y^2 - 1)\dot{y} + y = 0$$

$\alpha > 0$ measures the strength of the damping. We set $z = \dot{y}$ to convert this into a system:

$$\dot{y} = z$$

$$\dot{z} = -y - \alpha(y^2 - 1)z.$$

Let's look at this system when α is very small, then when α is very large. These two extremes help us understand the case when α is anywhere in between.

$\underline{\alpha \ll 1}$

If $\alpha = 0$, we have an undamped oscillator with Hamiltonian $H(y, z) = (y^2 + z^2)/2$. The orbits are clockwise circles in the phase plane around the center $y = z = 0$. If $\alpha \ll 1$, the orbits should still roughly be circles and the system is still nearly Hamiltonian. In fact

$$dH/dt = y\dot{y} + z\dot{z} = -\alpha(y^2 - 1)z^2$$

An orbit has $H = r^2/2 + O(\alpha)$, $y(t) = r\cos(t) + O(\alpha)$, $z(t) = r\sin(t) + O(\alpha)$. The radius r will change slowly on a timescale α^{-1} due to the small damping term. Substituting into the equation for dH/dt,

$$dH/dt = \alpha\{r^4\cos^2(t)\sin^2(t) - r^2\sin^2(t)\} + O(\alpha^2)\ .$$

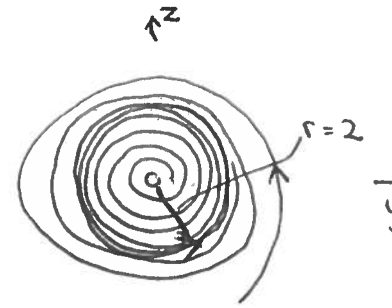
H fluctuates by $O(\alpha)$ in each cycle, but of more significance is the mean change averaged around a cycle in H, because it is this that slowly but systematically changes r. Letting a mean over one period 2π be denoted by $<\bullet>$, and noting that $<\sin^2(t)> = 1/2$, $<\cos^2(t)\sin^2(t)> = <\sin^2(2t)>/4 = 1/8$, we use the *Method of Averaging* in which we average H around one cycle:

$$d<H>/dt = \alpha\{r^4/8 - r^2/2\} + O(\alpha^2)$$

Thus

$$dr/dt = \alpha\{r^3/8 - r/2\} + O(\alpha^2).$$

If $r < 2$, r is slowly increasing (i. e. the orbits spirals slowly outward), and if $r > 2$, r is slowly decreasing The phase portrait is shown at right. $r = 2$ is a stable *limit cycle.* A limit cycle is a periodic orbit in a non-Hamiltonian system.



Poincare Maps

The Poincare map is a tool for analyzing periodically or roughly periodically varying systems and determining the stability of periodic orbits. Let Σ be a surface which cuts transversely to the orbits (i. e. Σ is not tangent to any orbit). For any point $\mathbf{u}_0$ on Σ, the Poincare map of $\mathbf{u}_0$ is the $\mathbf{u}(t)$ corresponding to the first intersection of the orbit based at $\mathbf{u}_0$ with Σ.

A Poincare map for our example above is easy to construct . We could choose Σ to be the positive x-axis, for instance. There is nothing special about this choice except that it intersects each orbit exactly once per revolution; the positive y axis would have worked just as well, but the full x axis intersects each orbit *twice* per revolution so would not be usable. Consider an orbit based on (starting at) the point $(x_0 , 0)$, so that $r(0) = x_0$ in the example above. After a time very nearly equal to 2π ,the orbit again intersects the positive x-axis at a new radius

$$x_1 \approx x_0 + 2\pi(dr/dt)(x_0) = x_0 - \pi\alpha(x_0{}^3/4 - x_0), \quad \alpha << 1$$

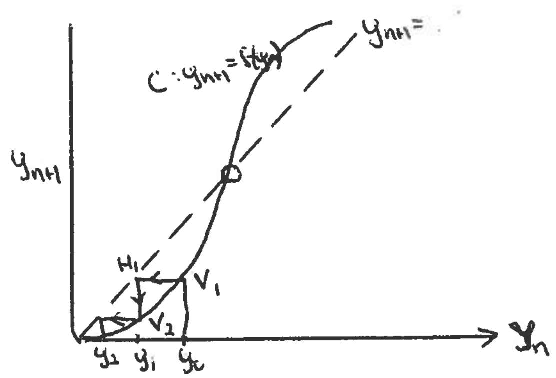
After another time 2π, the orbit will again intersect the positive x - axis at a position x_2 which is related to x_1 by this same formula. Similarly, the n+1'st intersection of the orbit with the positive x-axis is related to the n'th by this same nonlinear difference equation:

$$x_{n+1} \approx x_n + 2\pi(dr/dt)(x_n) = x_n - \pi\alpha(x_n{}^3/4 - x_n), \quad \alpha << 1 \qquad (*)$$

This is the Poincare map of the positive x-axis for this system. Since the positive x-axis is one dimensional, it is called a *one-dimensional map*. One dimensional maps come up frequently in studying systems of nonlinear ODE's and are very easy to analyze; we digress to show how this is done.

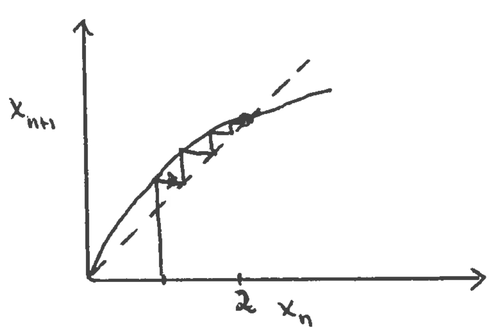
One-Dimensional Maps

An autonomous first order difference equation of the form $y_{n+1} = f(y_n)$ is called a *one-dimensional* (1D)*map*. It is best analyzed by plotting the curve C: $y_{n+1} = f(y_n)$ against y_n, and also plotting a straight line L: $y_{n+1} = y_n$ of slope 1 on the same graph. The *fixed points* of the map are the points where $y_n = f(y_n)$, i.e. the two curves intersect.



The iterates $y_1 = f(y_0), y_2 = f(y_1)\ldots$ for a given initial condition y_0 can be found graphically by starting on the horizontal axis at y_0 then drawing a vertical line up to its intersection V_1 with C; the vertical coordinate of this point is $y_1 = f(y_0)$. Next we draw a horizontal line until it intersect L at a point H_1 (which has coordinates (y_1, y_1), then draw another vertical line from H_1 to its intersection V_2 with C; the vertical coordinate of this point is $y_2 = f(y_1)$. We then keep repeating the process. By considering graphicallywhat happens to initial conditions which are near fixed points of the map, it is easy to see that a fixed point is stable if $|f\,'(y_0)| \leq 1$ and unstable otherwise. If $f\,'(y_0) > 0$ the approach to the fixed point is monotonic; if $f\,'(y_0) < 0$ the approach to the fixed point is oscillatory.

Application to the Poincare Map of the Example



The Poincare map (*) has the graph above. There is a fixed point $x_0 = 2$; this corresponds to a periodic orbit of the ODE since the initial condition is exactly returned to after one revolution. The fixed point is stable and has $f'(x_0) > 0$, so the orbits spiral monotonically toward the periodic orbit . Note that the stability analysis of the periodic orbit has reduced to a local stability analysis of a Poincare map, a distinct conceptual convenience because it reduces the dimensionality of the problem by 1. However, in general, Poincare maps must be found numerically since3 they require integrating a nonlinear system of equations until an orbit starting on Σ returns to Σ, which is rarely possible analytically (even in the example our analysis was approximate and only accurate in the limit $\alpha << 1$.

<u>$\alpha \gg 1$</u>

If $\alpha \gg 1$the system is dominated by the the damping term. Let's first attempt to sketch the phase portrait. Recall the equations:

$$\dot{y} = z$$

$$\dot{z} = -y - \alpha(y^2 - 1)z.$$

The only fixed point is $z = y = 0$. The linearized system around this point is

$$\dot{\xi} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix}}_{DF(0)} \xi, \quad \xi = \begin{bmatrix} y \\ z \end{bmatrix}$$

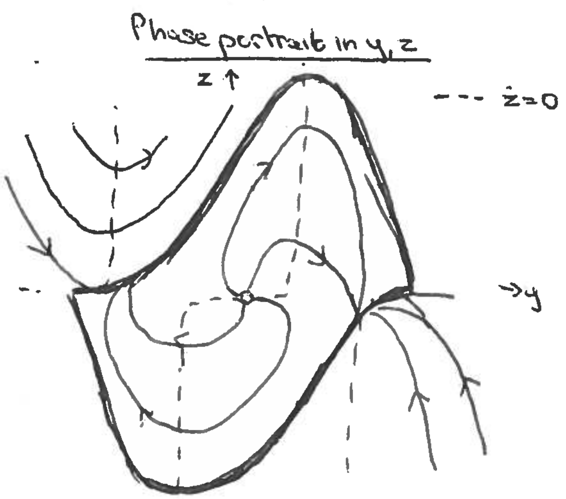
The eigenvalues are given by $\lambda(\lambda - \alpha) + 1 = 0$. For large α they are approximately α and α^{-1}; $\mathbf{0}$ is an unstable node. Following the method of isoclines, we see $\dot{z} = 0$ (the z *nullcline* (from 'no slope')) $y + \alpha(y^2 - 1)z = 0$. To a first approximation the nullclines are just given by $(y^2 - 1)z = 0$, or $z = 0$ and $y = \pm 1$. Elsewhere, the sign of $\dot{z}$ is opposite to the sign of

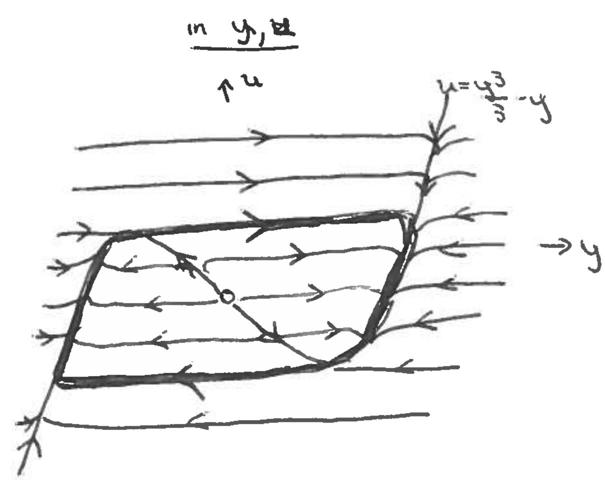
$+\alpha(y^2-1)z$ and $dz/dy|_{\text{orbit}} = \dot{z}/\dot{y} \approx -\alpha(y^2-1)$ is very large. This gives the phase portrait below at left. It is not immediately clear what happens near $z = 0$ for $y^2 > 1$.

Conceptually it is easier to think about the system and resolve this question if we straighten out the rapidly changing parts of the orbits into horizontal lines. Integrating the approximate expression for $dz/dy|_{\text{orbit}}$, we see $z \approx -\alpha(y^3/3 - y)$ + constant on orbits. Hence, if we use a new variable $u = z/\alpha + (y^3/3 - y)$ in place of z the rapidly varying part of the orbits should be $u \approx$ constant. The system can be written in terms of u and y as follows:

$$\dot{y} = z = \alpha\{u - (y^3/3 - y)\} \; .$$

$$\dot{u} = \dot{z}/\alpha - \dot{(y^3/3 - y)} = \ddot{y}/\alpha - (y^2\text{-}1)\dot{y} = -y/\alpha \; .$$





The phase portrait in u and y, shown above right, is more transparent. The y nullcline is $u = y^3/3 - y$. y changes very quickly at a rate $O(\alpha)$ away from this nullcline, while u changes very slowly at a rate $O(\alpha^{-1})$, so the orbits away from the y-nullcline are approximately horizontal. Thus to first order we can look at the equation for y alone, treating u as a constant. It has fixed points on the nullcline and they are stable if $\partial\dot{y}/\partial y = -\alpha(y^2 - 1) < 0$, i. e. $y^2 > 1$. All orbits are very rapidly attracted to a limit cycle in which the orbit slowly (in a time $O(\alpha)$) moves up the left stable branch of the nullcline, since u slowly increases for $y < 0$ (where $-y/\alpha > 0$). At $y = 1$ it flips rapidly (in a time $O(\alpha^{-1})$) over to the other branch and slowly moves down it, then flips over to the first branch to complete the cycle. Such periodic or nearly periodic orbits in which slow changes are punctuated by rapid jumps are called *relaxation oscillations*.