On the Effects of Moisture on the Brunt-Väisälä Frequency

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ABSTRACT

Expressions are derived for the Brunt-Väisälä frequency N_m, in a saturated atmosphere, which are analogous to commonly-used formulas for the dry Brunt-Väisälä frequency. These formulas are compared with others which have appeared in the literature, and the derivation by Lalas and Einaudi (1974) is found to be correct. The simplifying assumptions, implicit in derivations by Dudis (1972) and Fraser *et al.* (1973) are clarified. Numerical examples are presented which suggest that these incomplete formulations for N_m are reasonably accurate approximations, except when the saturated static stability is small. A new formula expressing N_m in terms of moist conservative variables is presented, and an accurate approximation is also given which may be useful when evaluating N_m.

1. Introduction

The Brunt-Väisälä frequency is one of the most widely used parameters in dynamic meteorology. According to its conceptual definition, it is the frequency at which an air parcel will oscillate when subjected to an infinitesimal perturbation in a stably stratified atmosphere. As a measure of the strength of the buoyancy force, it also appears in various nondimensional numbers in fluid mechanics, including the Richardson number, the Rayleigh number and the Froude number. In unsaturated air the Brunt-Väisälä frequency N may be calculated either from the expression

$$N^2 = \frac{g}{T}\left(\frac{dT}{dz} + \Gamma_d\right), \tag{1a}$$

or

$$N^2 = g\frac{d\ln\theta}{dz}, \tag{1b}$$

where T and θ are the sensible and potential temperatures of the atmosphere, Γ_d is the dry adiabatic lapse rate, and g is the gravitational acceleration.

When the atmosphere is saturated, the effective Brunt-Väisälä frequency N_m changes. If a parcel of air is displaced slightly upward in an unsaturated atmosphere, it will cool by adiabatic expansion. If the atmosphere is saturated, upward displacements are also accompanied by condensation and latent heating. This heating partially compensates for the cooling produced by adiabatic expansion, so the density difference between the parcel and the surrounding air is less than in the dry case, and the buoyancy restoring force is decreased. The absorption of latent heat due to evaporation in downward displaced parcels produces a similar reduction in the buoyancy restoring force. Thus the effective Brunt-Väisälä frequency is lower when the atmosphere is saturated than when it is dry. Indeed, it is not uncommon for the atmosphere to be stable when dry, and unstable when saturated (conditional instability). This stability reduction can be quantified by deriving an expression for the saturated Brunt-Väisälä frequency. Dudis (1972), Fraser *et al.* (1973) and Lalas and Einaudi (1974) have presented different expressions for N_m. The differences among their results do not seem to be widely recognized and, perhaps as a result, there are no widely accepted equivalents to (1) which define the saturated Brunt-Väisälä frequency.

One might be tempted to estimate N_m by replacing θ, in (1b), with a moist-conservative variable such as the equivalent potential temperature θ_e, to obtain

$$N_m{}^2 = g\frac{d\ln\theta_e}{dz}. \tag{2}$$

One might also expect to obtain the correct expression by replacing the dry adiabatic lapse rate in (1a), with the saturated adiabatic lapse rate Γ_m, in which case

$$N_m{}^2 = \frac{g}{T}\left(\frac{dT}{dz} + \Gamma_m\right). \tag{3}$$

However, as noted by Fraser *et al.* (1973), (2) and (3) are not equivalent. Fraser *et al.* accept (3) as the correct expression for the moist Brunt-Väisälä frequency. Dudis (1972) has suggested that an additional

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factor is required, so that

$$N_m^2 = \frac{g}{T}\left(\frac{dT}{dz} + \Gamma_m\right)\left(1 + \frac{Lq_s}{RT}\right), \qquad (4)$$

where q_s is the saturation mixing ratio, L is the latent heat of vaporization and R is the ideal gas constant for dry air. In fact, the correct formulation, given by Lalas and Einaudi (1974), contains an additional term,

$$N_m^2 = \frac{g}{T}\left(\frac{dT}{dz} + \Gamma_m\right)\left(1 + \frac{Lq_s}{RT}\right) - \frac{g}{1 + q_w}\frac{dq_w}{dz}, \qquad (5)$$

where q_w, the total water mixing ratio, is the sum of q_s and the liquid water mixing ratio q_L. Fraser *et al.* did not obtain the extra terms because they did not include virtual temperature effects in the buoyancy term of the vertical momentum equation. Dudis overlooked the term involving the total water mixing ratio by assuming that any increase in water vapor would be accompanied by an equal decrease in liquid water, i.e., $q'_w = 0$. While this is true following a parcel trajectory, it does not apply to equations written in an Eulerian coordinate system, for which the correct linearized equation for the conservation of total water is

$$\frac{\partial q'_w}{\partial t} + \bar{u}\frac{\partial q'_w}{\partial x} + w'\frac{d\bar{q}_w}{dz} = 0. \qquad (6)$$

Perturbation and mean state variables are denoted by primes and overbars, respectively.

The significance of the various terms which comprise expression (5) are not generally understood, perhaps because the derivations are complicated and sometimes hard to follow. In Section 2, we attempt to explain the origin of these terms by rederiving (5) via a simple parcel theory argument. In a dry flow it is often useful to express N in terms of the conservative variable θ. However, no corresponding form exists which expresses N_m correctly, in terms of moist conservative variables. Therefore, in Section 3, we derive an expression for N_m which is analogous to the form (1b). Finally, in Section 4, we compare the various formulations for N_m in two numerical examples, and suggest an approximate expression for N_m which is accurate and easier to compute.

2. Parcel theory

As noted by Lalas and Einaudi (1974), the effect of latent heat release on a disturbance in a saturated atmosphere depends on the ratio of the characteristic time scale of the disturbance to the time scale of the condensation processes. In the following, we will assume that this ratio is large, so that the atmosphere is always just at saturation. We consider only saturated moist adiabatic flow, assuming that the liquid water concentration in the environment is low, so that the water is distributed in very small droplets which do not precipitate and do not exert an inertial drag on the air motions. This assumption is appropriate for small amplitude motions which produce correspondingly small amounts of condensation. Precipitation will also be absent in larger amplitude perturbations if the time scale for flow through the wave is less than the time required for the growth of large water droplets.

The acceleration, due to buoyancy forces, experienced by a parcel displaced an infinitesimal distance δ from its equilibrium level is

$$a = -g\left(\frac{\rho_p - \rho_e}{\rho_e}\right), \qquad (7)$$

where ρ is the total density and the subscripts p and e denote values for the parcel and the environment. The densities may be expanded in a Taylor series about their equilibrium values at $\delta = 0$, where $\rho_p(0) = \rho_e(0)$. Retaining the terms through $O(\delta)$ we obtain

$$a = N_m^2\delta,$$

where

$$N_m^2 = g\left\{\left(\frac{d\ln\rho}{dz}\right)_p - \left(\frac{d\ln\rho}{dz}\right)_e\right\} = g\left.\frac{d\ln\rho}{dz}\right|_e^p. \qquad (8)$$

The equation of state for saturated air can be written as

$$p = \rho RT\frac{[1 + (q_s/\epsilon)]}{(1 + q_w)}, \qquad (9)$$

where $\epsilon = R/R_v$; R_v is the gas constant for water vapor. Substituting for ρ in (8) from (9), and assuming that the pressure in the parcel adjusts instantly to that in the environment, one obtains

$$N_m^2 = g\left.\left\{\frac{1}{T}\frac{dT}{dz} + \frac{1}{\epsilon + q_s}\frac{dq_s}{dz} - \frac{1}{1 + q_w}\frac{dq_w}{dz}\right\}\right|_p^e. \qquad (10)$$

Note that if virtual temperature effects had been ignored, (10) would be identical to (3). Assuming instantaneous adjustment to moist equilibrium, q_s can be evaluated via the Clausius-Clapeyron equation

$$\frac{1}{e_s}\frac{de_s}{dT} = \frac{\epsilon L}{RT^2}, \qquad (11)$$

where e_s is the partial pressure of water vapor at saturation. Thus

$$\frac{dq_s}{dz} = \frac{d}{dz}\left(\frac{\epsilon e_s}{p - e_s}\right) = \left(1 + \frac{q_s}{\epsilon}\right)\left(\frac{\epsilon Lq_s}{RT^2}\frac{dT}{dz} - \frac{q_s}{p}\frac{dp}{dz}\right). \qquad (12)$$

We then substitute from (12) for q_s in (10), and again invoke the assumption that the pressure in the parcel adjusts to that in the environment. Since the total water mixing ratio is constant inside the parcel, and

$(dT/dz)_p = -\Gamma_m$, one obtains

$$N_m^2 = \frac{g}{T}\left(\frac{dT}{dz} + \Gamma_m\right)\left(1 + \frac{Lq_s}{RT}\right) - \frac{g}{1+q_w}\frac{dq_w}{dz}. \quad (13)$$

Thus the saturated Brunt–Väisälä frequency is given correctly by (5). The expression (4) would be obtained by ignoring the gradient of the total water mixing ratio in the environment. Note that this result is independent of the details of the thermodynamics since the derivation did not require any use of the moist thermodynamic equation.

3. Conservative variable form for N_m

When calculating the dry Brunt–Väisälä frequency, the potential temperature formulation (1b) is usually more useful than the expression (1a), involving the difference between the dry adiabatic and environmental lapse rates. In the following, we will derive a corresponding expression for N_m using moist-conservative variables.

The classical reversible saturation adiabat may be written

$$d(\ln\theta_d) + \frac{L}{c_pT}dq_s + \left(\frac{c_{pv}q_s + c_wq_L}{c_p} - \frac{Lq_s}{c_pT}\right)d(\ln T) = 0. \quad (14)$$

Here, c_p, c_{pv} and c_w are the heat capacities of dry air, water vapor and liquid water and θ_d is the dry potential temperature, defined by

$$\theta_d = T\left(\frac{P_d}{P_0}\right)^{-R/c_p},$$

where P_d is the partial pressure of dry air and P_0 = 1000 mb. The form (14) can be derived from the standard expression for the reversible saturation adiabat (see Iribarne and Godson, 1973), using the relation

$$dL = (c_{pv} - c_w)dT. \quad (15)$$

A moist-conservative variable associated with (14) is

$$\theta_q = \theta_e\left(\frac{T}{T_0}\right)^{c_wq_w/c_p}, \quad (16)$$

where T_0 is a reference temperature and θ_e is the equivalent potential temperature given by

$$\theta_e = \theta_d \exp\left(\frac{Lq_s}{c_pT}\right). \quad (17)$$

This expression differs slightly from the wet equivalent potential temperature

$$\theta_q^* = T\left(\frac{P_d}{P_0}\right)^{-R/(c_p+c_wq_w)} \exp\left\{\frac{Lq_s}{(c_p+c_wq_w)T}\right\}$$
$$\propto \theta_q^{\,c_p/(c_p+c_wq_w)}$$

(see Paluch, 1979). Since q_w is conserved following a parcel, both θ_q and θ_q^* are conserved during reversible saturated adiabatic motion, i.e., by Eq. (14); however, the use of θ_q leads to a simpler expression for N_m^2. Note also that both θ_q and θ_q^* differ from θ_e by a small $O(q_w)$ term.

The moist adiabatic lapse rate, $\Gamma_m = -dT/dz$, can now be determined from (14). If the air motions are confined to vertical displacements,

$$\frac{d\ln\theta_d}{dz} + \frac{L}{c_pT}\frac{dq_s}{dz} - \left(\frac{c_{pv}q_s + c_wq_L}{c_p} - \frac{Lq_s}{c_pT}\right)\frac{\Gamma_m}{T} = 0. \quad (18)$$

Eliminating dq_s/dz using (12), and vertical pressure gradients via the hydrostatic equation, one obtains the following expression for the moist adiabatic lapse rate

$$\Gamma_m = \Gamma_d(1+q_w)\left(1 + \frac{Lq_s}{RT}\right) \times \left\{1 + \frac{c_{pv}q_s + c_wq_L}{c_p} + \frac{\epsilon L^2q_s}{c_pRT^2}\left(1 + \frac{q_s}{\epsilon}\right)\right\}^{-1}. \quad (19)$$

The dry adiabatic lapse rate may be written

$$\Gamma_d = \frac{g}{c_p} = -\frac{RT}{c_p}\frac{[1 + (q_s/\epsilon)]}{(1+q_w)}\frac{d\ln p}{dz}. \quad (20)$$

We substitute the expressions for Γ_m and Γ_d [(19) and (20)] into (5), and obtain the desired expression for N_m by using (12) and the definition of θ_q (16). Then

$$N_m^2 = \frac{g}{1+q_w}\left\{\frac{\Gamma_m}{\Gamma_d}\frac{d\ln\theta_q}{dz} - \frac{dq_w}{dz}\right\}. \quad (21)$$

If, in the thermodynamic equation, the heat absorbed by the water had been neglected, (14) would require the exact conservation of the equivalent potential temperature (17), and θ_e would replace θ_q in (21).

As is the case with the dry equations of motion, the use of conservative variables in the thermodynamic equation and Brunt–Väisälä frequency can greatly simplify the mechanics of many derivations involving the linearized equations of saturated-adiabatic motion. As an example, consider the problem of rescaling the linearized equations of motion to remove the effects of the decrease in mean density with height. In the dry case, all the mean state variables in the resulting system can be expressed in terms of N and c, where c is the speed of sound (see for example Bretherton, 1966). An identical system can be obtained by a similar scaling of the saturated equations, as demonstrated below.

The linearized equations of motion, continuity and thermodynamics for a saturated atmosphere may be written:

$$\frac{Du'}{Dt} + \frac{d\bar{u}}{dz}w' + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} - fv' = 0, \tag{22}$$

$$\frac{Dv'}{Dt} + \frac{d\bar{v}}{dz}w' + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} + fu' = 0, \tag{23}$$

$$\frac{Dw'}{Dt} + \frac{g\rho'}{\bar{\rho}} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} = 0, \tag{24}$$

$$\frac{1}{\bar{\rho}}\frac{D\rho'}{Dt} + \frac{d\ln\bar{\rho}}{dz}w' + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{25}$$

$$\frac{D\theta_q'}{Dt} + \frac{d\bar{\theta}_q}{dz}w' = 0, \tag{26}$$

where u, v and w are the velocity components in the Cartesian x, y and z directions, respectively, primes denote perturbation variables, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y}.$$

Together with (6), (9) and (11), these equations form a closed system.

We now introduce new variables such that

$$(u', v', w', \sigma') = \left(\frac{\bar{\rho}}{\rho_0}\right)^{-1/2}(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\sigma}), \tag{27a}$$

$$p' = \left(\frac{\bar{\rho}}{\rho_0}\right)^{1/2}\tilde{p}, \tag{27b}$$

where

$$\sigma' = \frac{-g}{1+\bar{q}_w}\left(\frac{\Gamma_m}{\Gamma_d}\frac{\theta_q'}{\bar{\theta}_q} - q_w'\right) = \frac{g\rho'}{\bar{\rho}} - \frac{gp'}{c_m{}^2\bar{\rho}}, \tag{28}$$

$$\left.\begin{aligned} c_m{}^2 &= c^2\left\{\frac{c_p}{c_v}(1+\bar{q}_w) - \frac{R}{c_v}\frac{\Gamma_m}{\Gamma_d}\left(1+\frac{L\bar{q}_s}{R\bar{T}}\right)\right\}^{-1}, \\ c^2 &= \frac{c_p}{c_v}R\bar{T}. \end{aligned}\right\} \tag{29}$$

In addition, define

$$H^{-1} = \frac{-g}{c_m{}^2} - \frac{1}{2}\frac{d\ln\bar{\rho}}{dz}. \tag{30}$$

Then Eqs. (22) to (26) become

$$\frac{D\tilde{u}}{Dt} + \frac{d\bar{u}}{dz}\tilde{w} + \frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial x} - f\tilde{v} = 0, \tag{31}$$

$$\frac{D\tilde{v}}{Dt} + \frac{d\bar{v}}{dz}\tilde{w} + \frac{1}{\rho_0}\frac{\partial\tilde{p}}{\partial y} + f\tilde{u} = 0, \tag{32}$$

$$\frac{D\tilde{w}}{Dt} + \tilde{\sigma} + \frac{1}{\rho_0}\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\tilde{p} = 0, \tag{33}$$

$$\frac{1}{c_m{}^2\rho_0}\frac{D\tilde{p}}{Dt} + \frac{\partial\tilde{u}}{\partial x} + \frac{\partial\tilde{v}}{\partial y} + \left(\frac{\partial}{\partial z} + \frac{1}{H}\right)\tilde{w} = 0, \tag{34}$$

$$\frac{D\tilde{\sigma}}{Dt} - N_m{}^2\tilde{w} = 0. \tag{35}$$

Eqs. (31) to (35) are identical to Bretherton's Eqs. (36) to (40), except that N_m and c_m have replaced N and c (Bretherton also neglects the Coriolis terms). Thus, two dynamical systems, with identical values of N and N_m, c and c_m, will have an identical linear behavior. As observed by Lalas and Einaudi (1973), the familiar analyses which produced expressions such as the wave dispersion relation or the critical Richardson number may be trivially modified for application to saturated conditions by replacing N with N_m, and c with c_m. A detailed discussion of the effects of saturation on gravity wave propagation is given in Einaudi and Lalas (1973), and a thorough treatment of the effect of moisture on the critical Richardson number appears in Lalas and Einaudi (1973).

The equivalence of the moist and dry systems described above is strictly applicable only in the linear case. The nonlinear behaviors differ primarily because the saturated adiabatic lapse rate Γ_m is not independent of height. As a parcel rises, the latent heat released per unit mass decreases as its vertical displacement increases. Thus, in a saturated gravity wave, nonlinear effects act to increase the buoyancy restoring force in the wave crests beyond that predicted by linear theory, while decreasing it in the troughs. The importance of this asymmetry depends on the amplitude of the wave. As demonstrated by Durran and Klemp (1982), the difference in the behavior of linearly equivalent wet and dry systems becomes significant in moderately strong waves.

One may interpret c_m as the speed of sound in a saturated atmosphere, in which the condensation processes occur on a much shorter time scale than one sound wave period. Clark and Hall (1979) have suggested that condensation and evaporation occur on a characteristic time scale of 1–5 s, so c_m will not give the correct speed of sound for frequencies in the audible range. Nevertheless, it may be correct for low frequency (<0.1 hz) infrasound disturbances.

4. Comparison of expressions for N_m

How important are the differences in the various expressions for the saturated Brunt–Väisälä frequency? Fig. 1a shows a plot of the square of the saturated Brunt–Väisälä frequency computed from several different formulations versus the sensible temperature lapse rate, when the atmospheric temperature is 25°C and the pressure is 1000 mb. The dry Brunt–Väisälä frequency is also plotted for reference. In order to evaluate $N_m{}^2$ we must also specify the vertical distributions of water vapor and liquid water. The water vapor concentration is determined by the saturation mixing ratio, but the liquid water distri-

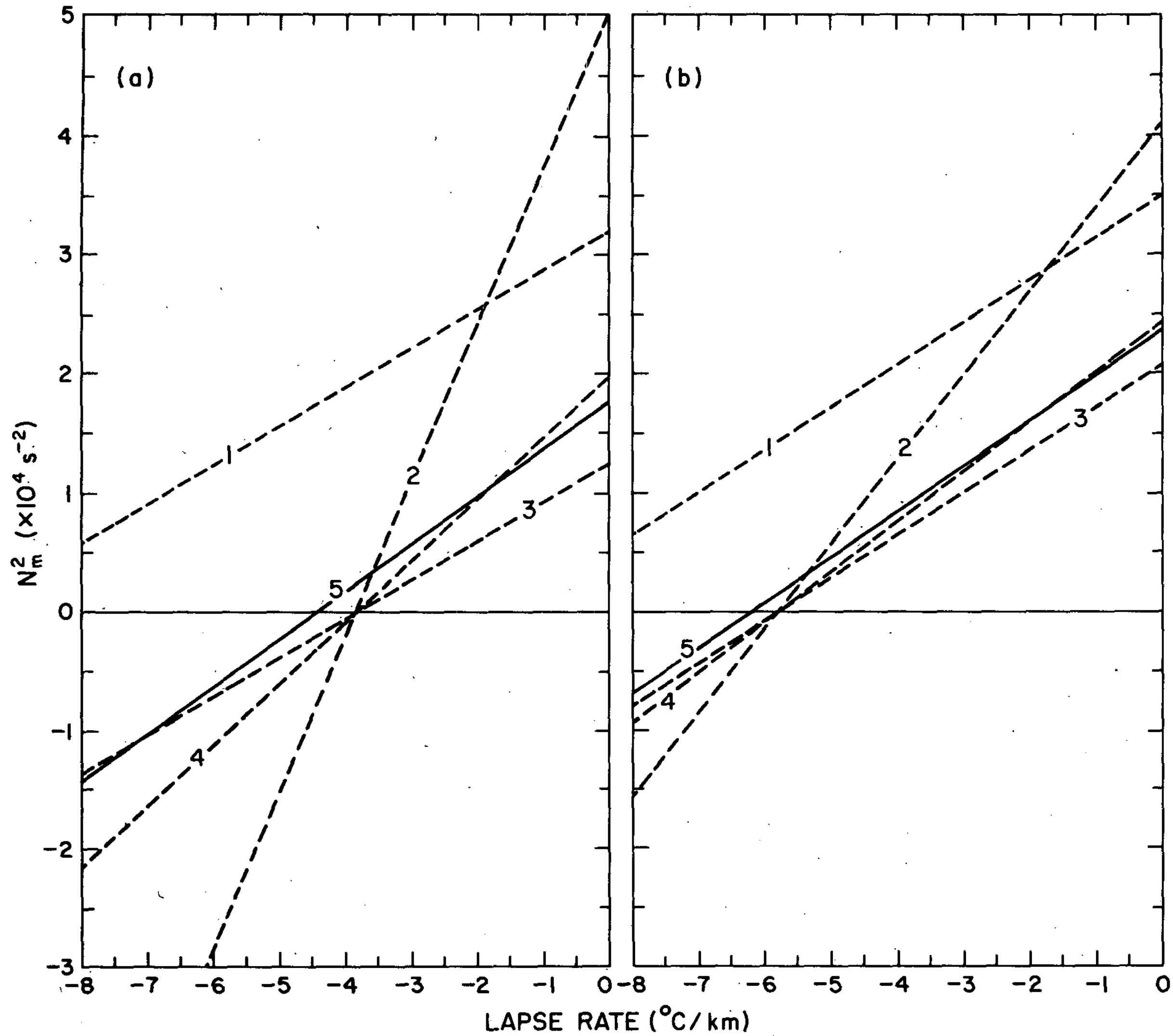

FIG. 1. The Brunt–Väisälä frequency, computed from expressions (1) through (5), as a function of the environmental lapse rate when the atmospheric temperature and pressure are: (a) 25°C and 1000 mb; (b) 0°C and 700 mb. The correct result, (5), is shown as a solid line.

bution depends on the history of the motion of individual air parcels. In these calculations, we arbitrarily specify $\partial \bar{q}_w/\partial z = \partial \bar{q}_s/\partial z$, which is a reasonable approximation in many realistic situations.

As expected, the actual saturated stability (5) is always much less than the dry stability (1). Formulations (3) and (4) produce reasonable approximations to the exact result, while (2) is clearly a very poor approximation. The large error in (2) is principally due to neglect of the factor Γ_m/Γ_d in (21). This factor may be as small as 0.4. A critical difference between (5) and the approximate expressions (3) and (4) lies in the position of the $N_m^2 = 0$ intercept. As noted by Lalas and Einaudi (1974), a stable saturated atmosphere might appear to be statically unstable if the vertical gradient of the total water is ignored. In this example, the error encountered in predicting the critical lapse rate at which the saturated atmosphere would be neutrally stable is ~0.7°C km^{-1}. If there is vertical shear, so that the dynamic stability is determined by the Richardson number, the error produced by (3) will be even more significant.

The various expressions for N_m^2 are compared again in Fig. 1b when the atmospheric temperature is 0°C and the pressure is 700 mb. The relatively warm conditions in Fig. 1a allow the air to hold a large, though not unrealistic, amount of water vapor (20 g kg^{-1}). In Fig. 1b the air is much colder (holding 5 g kg^{-1} of water vapor), so one might expect the effects of latent heat release to be weaker, and indeed the difference between N^2 and N_m^2 is reduced. The differences between the various formulations for N_m^2

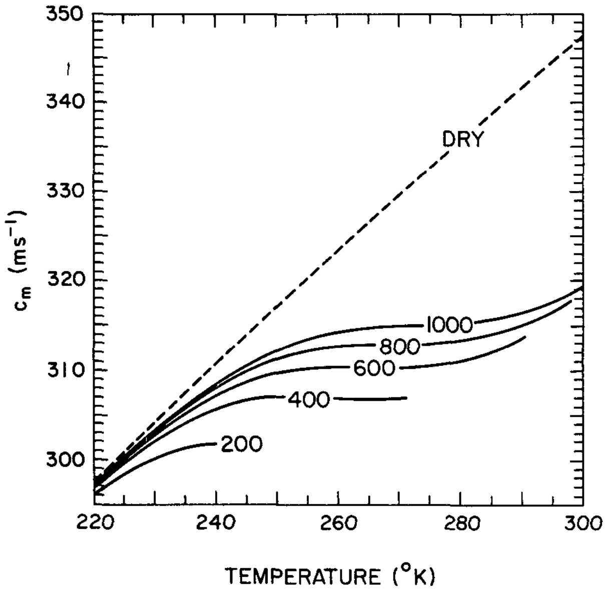

FIG. 2. The moist speed of sound c_m, versus temperature at ambient pressures between 200 and 1000 mb. The dry speed of sound is also plotted. The curves have been truncated to reflect the decrease in temperature with height.

are also decreased. However, (3) and (4) still overestimate the lapse rate at which the saturated atmosphere has neutral stability by almost 0.5°C per km.

We propose that a very good approximation for the saturated Brunt–Väisälä frequency may be obtained by simplifying (21) as follows

$$N_m^2 = g\left\{\frac{1 + (Lq_s/RT)}{1 + (\epsilon L^2 q_s/c_p RT^2)} \times \left(\frac{d\ln\theta}{dz} + \frac{L}{c_pT}\frac{dq_s}{dz}\right) - \frac{dq_w}{dz}\right\}. \quad (36)$$

This approximation has also been plotted in Fig. 1, and at the resolution shown in the figure, the result is indistinguishable from the exact expression.

This simplified expression for N_m^2 is obtained by using a simplified thermodynamic equation in the derivation of (21). Note that the reversible saturation adiabat (14) is expressed in terms of the potential temperature of the dry air, whereas (36) is written in terms of the usual potential temperature. Eq. (14) may be rewritten in terms of θ as

$$d(\ln\theta) + \left(1 + \frac{q_s}{\epsilon}\right)^{-1}\left\{\frac{L}{c_pT}dq_s + \left[\frac{c_w}{c_p}q_L + \left(\frac{c_{pv}}{c_p} - \frac{1}{\epsilon}\right)q_s\right]d(\ln T)\right\} = 0. \quad (37)$$

This form is identical to Eq. (8) derived by Lipps and Hemler (1980). Since the mixing ratios of water vapor and liquid water are always much less than one in atmospheric applications, one might expect to obtain a good approximation to (37) by neglecting all terms of $O(q_s)$ and $O(q_w)$. Then the approximate equation for a saturation adiabat becomes

$$d(\ln\theta) + \frac{L}{c_pT}dq_s = 0. \quad (38)$$

Expression (36), for N_m, is obtained by replacing (14) with (38), again omitting $O(q_s, q_w)$ terms as described above.

It is instructive to compare (38) with the equation for conservation of equivalent potential temperature, as defined by the widely accepted formula

$$\tilde{\theta}_e = \theta \exp\left(\frac{Lq_s}{c_pT}\right). \quad (39)$$

Note that θ_e is defined in terms of θ_d in (17), but $\tilde{\theta}_e$ is defined in terms of θ in (39). If we ignore the weak temperature dependence of L, $\tilde{\theta}_e$ is conserved when

$$d(\ln\theta) + \frac{L}{c_pT}dq_s - \frac{Lq_s}{c_pT}d(\ln T) = 0. \quad (40)$$

In comparing these approximate expressions, notice that (40) differs from (37) by terms which are $O(L/c_pT)$ larger than the error terms in (38). Since L/c_pT is $O(10)$, (40) produces a poorer approximation to N_m, and, at least for shallow convection, a poorer approximation to the saturation adiabat than (38). The reader is referred to Lipps and Hemler (1980) for a discussion of the importance of the distinction between θ and θ_d in various approximations to the thermodynamic equation for deep convection.

The effect of saturation on the speed of sound is shown in Fig. 2, in which c_m is plotted as a function of temperature at five pressure levels. The dry speed of sound is also plotted. As previously mentioned, c_m may be interpreted as the speed of sound in an atmosphere in which condensation processes occur on a much shorter time scale than one sound wave period. In the atmosphere, this criterion is only satisfied by low frequency infrasound waves. Inspection of Fig. 2 shows that the speed of these waves can be reduced by ~10% when they propagate through saturated regions at temperatures and pressures which support large saturation water vapor mixing ratios.

The significance of c_m is not limited to sound waves; it also characterizes the effects of compressibility on gravity waves. For instance, differentiating the equation of state (9), we find that the vertical gradient of the mean density in a hydrostatic atmosphere satisfies

$$\frac{d\ln\rho}{dz} = -\frac{N_m^2}{g} - \frac{g}{c_m^2}. \quad (41)$$

Thus, variations in c_m also affect the density scale height of gravity waves. Since the last term in (41) dominates, the mean perturbation amplitude of a linear disturbance (see Eq. 27) will increase more rapidly with height (by as much as 10%) when the atmosphere is saturated than when it is dry.

5. Conclusions

Three different expressions for the saturated Brunt–Väisälä frequency have appeared in the literature. We have compared these expressions and found the result of Lalas and Einaudi (1974) to be correct. Eq. (5) should be accepted as the saturated analog to (1a). Although correct, the alternative forms given by Lalas and Einaudi for the calculation of N_m are cumbersome to evaluate. We have derived an equivalent expression in terms of moist-conservative variables, and proposed (21) as the saturated analog of the dry form (1b). We have also presented a simpler approximation (36) which may be useful when actually computing N_m. The accuracy of the various expressions for N_m has been compared in two numerical examples. The approximation (36) appears to be very accurate. The other expressions in the literature, (3) and (4), produce reasonably accurate numerical results, but may cause significant error when the saturated static stability is small.

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Equation (21) should read

$$N_m^2 = \frac{g}{1+q_w}\left\{\frac{\Gamma_m}{\Gamma_d}\frac{d\ln\theta_q}{dz} - \left[1 + \frac{\Gamma_m}{\Gamma_d}\frac{c_w}{c_p}\ln\left(\frac{T}{T_0}\right)\right]\frac{dq_w}{dz}\right\}$$