GEOPHYSICAL RESEARCH LETTERS, VOL. ???, XXXX, DOI:10.1029/,

Refugium for surface life on Snowball Earth in a nearly-enclosed sea? A first simple model for sea-glacier invasion

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The existence of photosynthetic eukaryotic algae during the so-called Snowball Earth events presents a conundrum. If thick ice covered the oceans, where could such life persist? Here we explore the possibility that photosynthetic life persisted at the end of long narrow seas, analogous to the modern-day Red Sea. In this first analytical model, we test the ability of the global sea glacier to penetrate a Red Sea analogue only under climatic conditions appropriate during a Snowball Earth event. We find the Red Sea is long enough to provide a refugium only if certain ranges of climatic conditions are met. These ranges would likely expand if the effect of a narrow entrance strait is also considered.

1. Introduction

During the low-latitude glaciation of the Neoproterozoic, 600-800 Ma [*Evans*, 2000; *Harland*, 1964], the upper ocean may have become almost completely covered with thick ice, commonly known as Snowball Earth events [*Kirschvink*, 1992; *Hoffman et al.*, 1998; *Warren et al.*, 2002; *Pierrehumbert et al.*, 2011]. Photosynthetic eukaryotic algae did survive these events [*Knoll*, 1992; *Macdonald et al.*, 2010], indicating that liquid water was maintained at some locations at or near the ocean surface. In the face of evidence for global ocean ice cover, how did open water and life persist? With an equilibrium ice thickness (set by geothermal flux) reaching hundreds of meters, but thicker at high latitude than at low latitude, the floating ice would flow equatorward as a "sea glacier" [*Goodman and Pierrehumbert*, 2003], threatening to invade any unprotected areas of open water. Small pools of open water could be maintained above geothermal hotspots on coastlines of volcanic islands [*Hoffman and Schrag*, 2000], where the water is shallow enough that the geothermal heat is not diffused laterally and that the sea glacier becomes grounded and slowed by friction with the sea floor.

Here we investigate a possible class of refugia, isolated refuges for life, that could be much larger and also longer-lasting. A long narrow arm of a sea, for example as formed by continental rifting such as the modern Red Sea, located at low latitude and surrounded by desert land, would be invaded by a tongue of the global sea glacier, but this tongue would be impeded by friction with the sidewalls and thinned by sublimation, so that its thickness would diminish to zero before reaching the end of the sea, if the sea is long enough. This possibility was mentioned by *Warren et al.* [2002] and considered likely by *Pollard and Kasting* [2005, 2006]. The low-albedo (non-glaciated) land surrounding the bay at the inland end of the narrow sea might keep the bay warm enough to allow open water, even while the rest of the ocean was kept cold by its high albedo. In a GCM simulation of Snowball Earth, a rectangular continent centered on the equator did have higher average temperatures than the equatorial ocean, particularly at the eastern end of the continent [*Abbot and Pierrehumbert*, 2010].

In this paper we investigate only one aspect of this problem, namely the criterion for preventing the sea glacier from reaching the inland end of the sea. This is a necessary, but not sufficient, condition for the existence of a refugium. If the climate at the end of a narrow sea is too cold, the sea will fill with thick ice grown locally, even without any inflow from the global sea glacier. But even if the climate is warm enough to sustain open water or thin ice in the absence of ice flow, sea-glacier invasion could destroy the the refugium. To isolate and explore this latter threat, we consider temperature regimes that are otherwise favorable to a refugium. In this first exercise we further simplify the problem by considering only a sea of constant width W, which allows an analytical solution for the penetration distance L; we therefore are not modeling the narrow strait that commonly constricts the entrance to nearly-enclosed seas on the modern Earth, such as the Bab al Mandeb, the Strait of Gibraltar, and the Bosporus. In this model, the length-to-width ratio L/W at which the ice thickness diminishes to zero just at the far end of the narrow sea is a function of just three variables: the thickness H_0 at the entrance, the net sublimation rate at the top surface $\dot{b}$ (positive where sublimation exceeds precipitation), and the atmospheric temperature T_s (which determines the ice temperature and effective ice softness and thus the flow velocity). We assume that there is no melting or freezing at the base of the sea glacier tongue. If the climate in the refugium is warm enough to support open water, there might be melting at the surface or base of the glacier, which would reduce L and make the probability of a refugium more likely than our computations here suggest.

2. Why inland seas could protect refugia from sea glaciers

The flow of a sea glacier into a channel would be resisted by drag stress along the sidewalls of the channel. If this channel is located in a region of net sublimation, the inflowing sea glacier will lose mass as it invades the channel. Alternatively, if the narrow sea is in a region of net accumulation, it would be surrounded by glaciated land and filled with snow-covered ice; thus an inland sea can provide a refugium only if it is in a region of net sublimation, i.e., surrounded by extensive desert. If the channel is sufficiently narrow, the resistive drag stresses impede the inflowing sea glacier, and sublimation causes the sea glacier to shrink to zero thickness after some length L.

In order for an inland sea to provide a refugium for photosynthetic organisms during snowball events, several conditions must have been satisfied: (1) the inland sea must not be fully penetrated by a sea glacier; (2) the climate on the inland sea must be such as to maintain it either ice-free or covered by an ice layer sufficiently thin to allow photosynthesis below the ice; (3) the depth of the sea at its entrance, and throughout its length, must be great enough that seawater is able to flow under the sea glacier to replenish water loss from the refugium by evaporation/sublimation, and (4) water circulation in the inland sea must be adequate to allow nutrients to be delivered to organisms living in the bay at the landward end. Here, during this first exercise, we examine only the first condition.

3. Penetration length of sea glaciers

In order to determine whether inland seas could have remained free of sea-glacier ice, we estimate the penetration length L of a sea glacier flowing into an inland sea. The penetration length is defined as the distance a sea glacier, with an initial thickness H_0, would have traveled before reaching zero thickness. To determine the penetration length L we derive an analytical solution for flow along a parallel-sided channel using force balance and continuity. *Weertman* [1957] determined the creep rate for an unconfined ice shelf with a uniform thickness and width. *Thomas* [1973] expanded on the work of *Weertman* [1957] by considering the creep rate of a confined ice shelf. By assuming a yield stress for sidewall shearing, he concluded that velocity in the downstream direction is constant. *Sanderson* [1979] found that velocity in the downstream direction did vary, but the variation was small. Therefore, we will assume that velocity is uniform in the downstream direction.

Consider a sea glacier that enters a narrow channel with a uniform width W and vertical walls (Figure 1a). We choose the x-direction to be horizontal and parallel to the channel walls, the y-direction to be perpendicular to the channel walls, and the z-direction to be vertical, positive upward. The centerline of the sea glacier is at $y = 0$.

Environmental controls such as temperature and net rate of sublimation also control the penetration length of a sea glacier. In Snowball Earth models, zonally-averaged mean annual surface temperatures ranged from -50°C to -20°C at latitudes where net sublimation was likely [*Goodman*, 2006; *Pierrehumbert et al.*, 2011]. Ice is softer at warmer temperatures, hence a warmer sea glacier will flow faster so may penetrate farther. Basal temperatures were calculated by adjusting the melting point of ice for pressure and salt content. Salt content was estimated by assuming present-day total salinity and total volume of the ocean, and adjusting the salinity taking into account the salt-free ice covering the ocean during a Snowball Earth event. A relationship between ice temperature and the softness parameter $A(T)$ has been derived empirically [*Cuffey and Paterson*, 2010, pg. 75] and has a functional form

$$A(T) = A_0 exp\left(\frac{-Q_c}{RT}\right) \tag{1}$$

where A_0 represents a temperature-independent softness coefficient, Q_c represents the activation energy for creep, R represents the ideal gas constant, and T represents ice temperature. Values for these constants are given in Table 1. Because the rate of vertical advection of heat is small compared to the heat conduction rate, we assume a linear temperature profile throughout the ice thickness. Then we calculate an effective isothermal ice softness parameter A that satisfies force balance and produces the same ice flux in the channel as the value obtained by using the nonuniform temperature profile; a full explanation is given in the supplemental material. This new isothermal ice softness parameter A is used in place of the depth-dependent A in all equations that follow. Using an effective isothermal ice temperature to determine ice softness allows us to solve the problem using simpler isothermal equations.

The mass loss has two components, sublimation at the surface of the sea glacier, and melting at its base where it contacts ocean water. The net rate of sublimation, $\dot{b}$, is poorly constrained. In *Pierrehumbert*'s [2005] model (see also *Pierrehumbert et al.* [2011] Figure 7), the zonal average net sublimation rate peaked at the equator with a value of 5 mm/year, for 2000 ppm CO_2. During the coldest phase of a Snowball Earth event, CO_2 may have been just 100 ppm, and corresponding net sublimation of only 3 mm/year [*Pierrehumbert*, 2005, Figure 10]. *Goodman* [2006] calculated that mass loss through basal melting is insignificant compared to sublimation losses at equatorial latitudes. Therefore we use a parameter space to calculate penetration lengths where net rate of sublimation and ice softness are varied, while other parameters are held constant (Table 1).

3.1. Channel Velocity

Nye [1965] solved for the velocity for a glacier with infinite depth and uniform width W where flow is resisted by two parallel sidewalls. Nye's Equation 7 neglected vertical shearing because of the infinite thickness of the glacier. A sea glacier entering a narrow channel does not undergo vertical shearing because the sea glacier is floating and water cannot support a shear stress, so Nye's solution is appropriate:

$$u(x,y) = \frac{W}{2}\frac{A(x)k(x)^n}{n+1}\left(1 - \left|\frac{2y}{W}\right|^{n+1}\right) \tag{2}$$

where u represents velocity in the x-direction, k represents resistive drag stress at the walls, A represents ice softness, and n represents the exponent in the Glen ice flow law [*Glen*, 1955]. Nye's solution assumes no variation of velocity in the x-direction, but variation of u in the x-direction is permitted if k varies as a function of x. Variation of k in the

z-direction is not permitted, because there is no basal shear stress to cause a vertical gradient of velocity. Equation 2 can be integrated across the channel to calculate a mean velocity, $\bar{u}(x)$

$$\bar{u}(x) = \frac{W}{2}\frac{A(x)k(x)^n}{n+2} \qquad (3)$$

3.2. Continuity

We use the time-independent mass conservation equation

$$\nabla \cdot (\bar{\mathbf{u}}H) + \dot{b} = 0, \qquad (4)$$

where $\bar{\mathbf{u}}$ represents the ice-velocity vector, H represents local ice thickness, and $\dot{b}$ represents the net sublimation rate at the upper surface ($\dot{b}$ is positive for loss in volume). Motion is entirely in the x-direction, so Equation 4 can be written as

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}H(x) + \bar{u}(x)\frac{\mathrm{d}H}{\mathrm{d}x} = -\dot{b}(x) \qquad (5)$$

Substituting Equation 3 into Equation 5 results in

$$H(x)\frac{W}{2}\frac{A(x)}{n+2}\frac{\mathrm{d}}{\mathrm{d}x}(k(x)^n) + \frac{W}{2}\frac{A(x)k(x)^n}{n+2}\frac{\mathrm{d}H}{\mathrm{d}x} = -\dot{b}(x) \qquad (6)$$

3.3. Force Balance

To solve Equation 6, we must determine the resistive drag stress k as a function of x. We use a force-balance approach where we assume that the pressure gradient forces in a sea glacier flowing along a narrow channel are entirely balanced by the resistive drag force at the sides; a full explanation is given in the supplemental material, where Equation S-12 shows that the resistive drag stress at the walls is

$$k(x) = -\frac{W}{2}\Gamma\frac{\mathrm{d}H}{\mathrm{d}x} \tag{7}$$

where Γ is defined as

$$\Gamma \equiv \rho_i g \left(1 - \frac{\rho_i}{\rho_w}\right) \tag{8}$$

and ρ_i represents the density of ice, ρ_w represents the density of sea water, and g represents the acceleration due to gravity. By substituting Equation 7 into Equation 6, we can write a differential equation in terms of H

$$\frac{W}{2}\frac{A(x)}{n+2}\left(\frac{-W}{2}\Gamma\right)^n \left[nH(x)\left(\frac{\mathrm{d}H}{\mathrm{d}x}\right)^{n-1}\frac{\mathrm{d}^2H}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}H}{\mathrm{d}x}\right)^{n+1}\right] = -\dot{b}(x) \tag{9}$$

Equation 9 with the boundary condition $H(0) = H_0$ has a solution in which H varies linearly with x:

$$H(x) = H_0\left(1 - \frac{x}{L}\right) \tag{10}$$

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{H_0}{L} \tag{11}$$

$$\frac{\mathrm{d}^2H}{\mathrm{d}x^2} = 0 \tag{12}$$

The ratio L/W can be found by substituting Equations 10, 11, and 12 into Equation 6 to get

$$\frac{L}{W} = \frac{H_0}{D} \tag{13}$$

where

$$D = 2\left(\frac{\dot{b}(n+2)}{A\Gamma^n}\right)^{\frac{1}{n+1}} \tag{14}$$

is a characteristic length that depends on temperature, through A, and net sublimation rate $\dot{b}$.

Mean velocity, $\bar{u}$, across the channel can be calculated as

$$\bar{u} = \frac{W}{2}\left(\frac{A(\Gamma\dot{b})^n}{n+2}\right)^{\frac{1}{n+1}} \tag{15}$$

and flux, Q_0, entering the channel through the upstream end from the global sea glacier can be calculated as

$$Q_0 = WH_0\bar{u}. \tag{16}$$

4. Application to a modern analogue

Penetration length scales linearly with channel width (Equation 13); therefore, penetration length L can be non-dimensionalized in terms of channel width W. Figure 2 demonstrates that a sea glacier can penetrate farther into a narrow channel if it has lower net sublimation rate $\dot{b}$, and/or warmer air temperatures represented by larger ice softness parameter A.

As a characteristic inland sea we use the Red Sea, a nearly enclosed basin in the Red Sea Rift, a spreading center between the African and Arabian plates. Because the rifting of continental plates is caused by convection in the Earth's mantle, which was presumably not significantly affected by Snowball Earth events, it is reasonable to assume that inland seas like the Red Sea existed during Snowball Earth events.

To determine the penetration length of a sea glacier moving into a Red Sea analogue, we approximate the sea by a rectangle 200 km wide and 1300 km long; the length-to-width ratio L/W is 6.5. We assume that the upstream thickness H_0 of a sea glacier at the mouth of a Red Sea analogue is 650 m, consistent with ice thickness of the global sea glacier in regions of net sublimation [*Goodman*, 2006]; however, penetration length scales linearly with H_0 (Equation 13) so a thicker global sea glacier could penetrate proportionally farther. The length of the Red Sea exceeds the penetration length of the sea glacier if the combination of ice temperature and sublimation rate falls to the left of the dashed line in Figure 2. The along-channel ice thickness $H(x)$, and ice flux $Q(x)$ for a sea glacier decrease linearly with distance (Figure 1a-b) while mean along-channel velocity $\bar{u}$ remains constant.

The planetary climate of a Snowball Earth event warms over time through either the build-up of atmospheric CO_2 or changes in planetary albedo [*Abbot and Pierrehumbert*, 2010]. As the planet warms, the length-to-width ratio L/W of a sea glacier invading a narrow channel may change. We explore changes in L/W by exploring changes in the local net sublimation rate $\dot{b}$ and ice softness A due to a change in atmospheric temperature

T_s. Here, we assume that changes $\dot{b}$ are proportional to changes in the saturation vapor pressure from a Clausius-Clapeyron relation

$$\dot{b}(T_s) = c\ exp\left(\frac{-G}{T_s R}\right) \tag{17}$$

where c represents a constant based on initial conditions, and G represents the Clausius-Clapeyron exponent from *Marti and Mauersberger* [1993]. Atmospheric temperature T_s changes slowly enough to allow the sea glacier to always be in an equilibrium steady-state because the time scale of thermal diffusion in the penetrating sea glacier is much less than the time scale for global changes in T_s. Therefore T_s can be used as a proxy for time. Figure 3a shows an example calculation with initial climate conditions of T_s = -50°C and $\dot{b}$ = 1mm/year. The length-to-width ratio L/W of the sea glacier, prescribed by Equation 13, decreases in a warming world because increases in length due to increased ice softness are smaller than decreases in length due to increased net sublimation (Figure 3b). Both sublimation and shearing require breaking of hydrogen bonds. Correspondingly, the Clausius-Clapeyron exponent (51 kJ/mol) is similar to the activation energy for creep (60 kJ/mol); each is approximately the energy required to break two hydrogen bonds. The reason that A increases more slowly than $\dot{b}$ in Figure 3b is that $\dot{b}$ is determined by the surface temperature whereas A is determined by an effective depth-averaged temperature including the influence of warm ice at the bottom (-2°C) for all values of the surface temperature. Although this is a specific example, other reasonable initial conditions produce qualitatively similar results. In addition, this approach does not take into account surface and basal melting processes that would be expected once atmospheric conditions are near the melting temperature, and furthermore the upstream thickness H_0 of the global sea glacier entering the channel would decrease in a warming world. Both of these effects would decrease L/W for an invading sea glacier and L/W would be less than its initial value throughout the deglaciation.

5. Conclusions

As modeled here, an analogue to the Red Sea is long enough to have provided a refugium for photosynthetic organisms during Snowball Earth events, only if certain surface temperature and net sublimation conditions were met. The penetration length of an invading sea glacier continuously decreases as the planet warms during a Snowball Earth event. In future work, the restoring effect of a narrow entrance strait to the sea will be considered. That investigation will require a numerical model, but will likely allow a refugium to exist over a wider range of atmospheric temperatures and net sublimation rates.

Acknowledgments. This work was supported by NSF grant OPP-0739779. We thank John Booker for suggesting that we search for an analytical solution. We thank Matt Smith, Dorian Abbot, Raymond Pierrehumbert and an anonymous reviewer for their very helpful comments.

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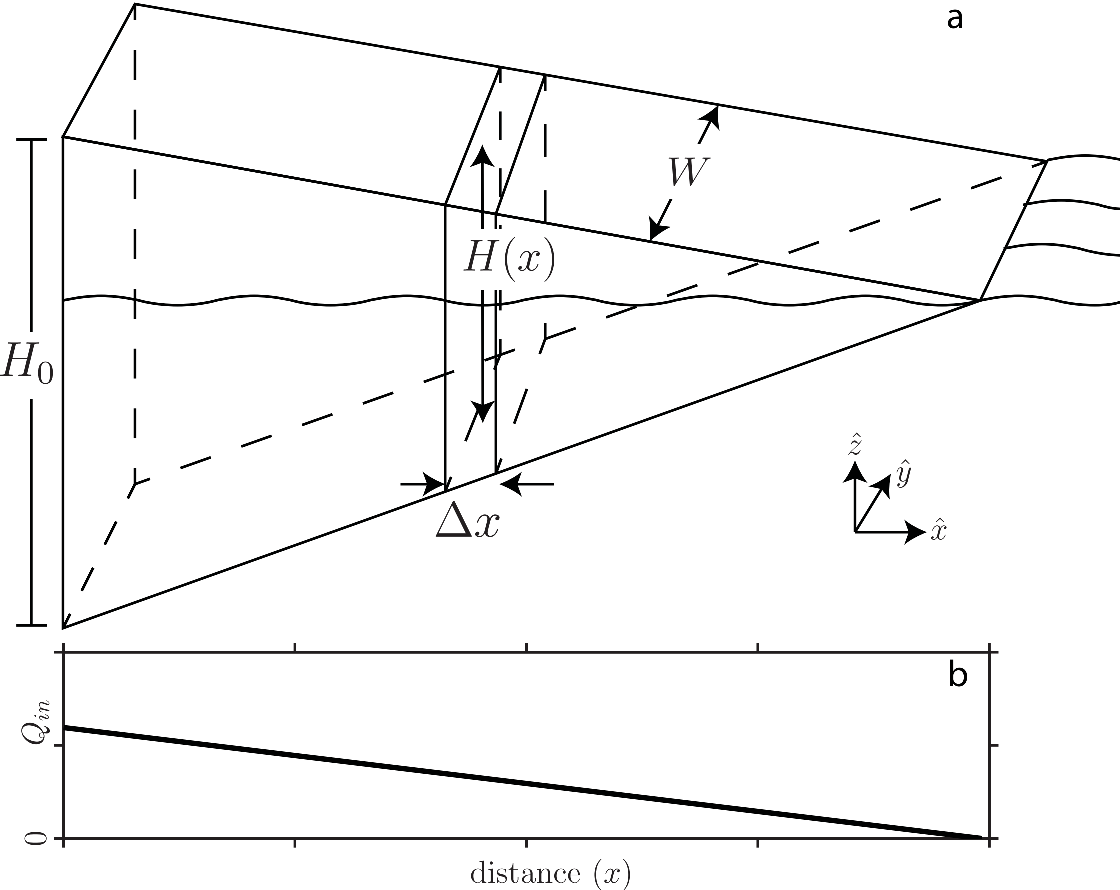
Figure 1. (a) Cartoon schematic of a sea glacier penetrating a channel. Ice thickness and ice flux (b) decrease linearly with distance, while mean across-channel velocity $\bar{u}$ is uniform along the channel.

Figure 2. Solid contours represent penetration length-to-width ratio L/W as a function of net sublimation rate $\dot{b}$ and surface ice temperature T_s for a sea glacier with an initial thickness H_0 = 650m, entering a narrow channel. Dashed line represents the 6.5 L/W ratio for the Red Sea. Atmospheric conditions to the left of the dashed line allow a refugium to avoid being over-ridden at the end of the Red Sea analogue.

Figure 3. Upper panel (a) shows how increasing atmospheric temperatures during deglaciation affect net sublimation rate $\dot{b}$, relative to an initial value $\dot{b}_0$ of 1 mm/year (dashed line), and ice softness A, relative an initial value A_0 set by a surface temperature T_s = -50°C (solid line). Net sublimation rate $\dot{b}$ is more sensitive than ice softness A to rising surface temperature T_s. Lower panel (b) shows resulting length-to-width ratio L/W of a invading sea glacier, calculated from Equation 13. For this particular example, the initial upstream ice thickness is H_0 = 650m.

Table 1. Constants used in analysis

Name	Symbol	Value	Units
temperature-independent ice softness parameter	A_0	4×10^{-13}	$Pa^{-3}\ s^{-1}$
Clausius-Clapeyron exponent	G	51	kJ/mol
acceleration of gravity	g	9.81	m/s^2
flow law exponent	n	3	dimensionless
activation energy for creep	Q_c	60	kJ/mol
ideal gas constant	R	8.314	$J\ mol^{-1}\ K^{-1}$
ice density	ρ_i	917	kg/m^3
seawater density	ρ_w	1043	kg/m^3


a
W
H(x)
H_0
Δx
ẑ
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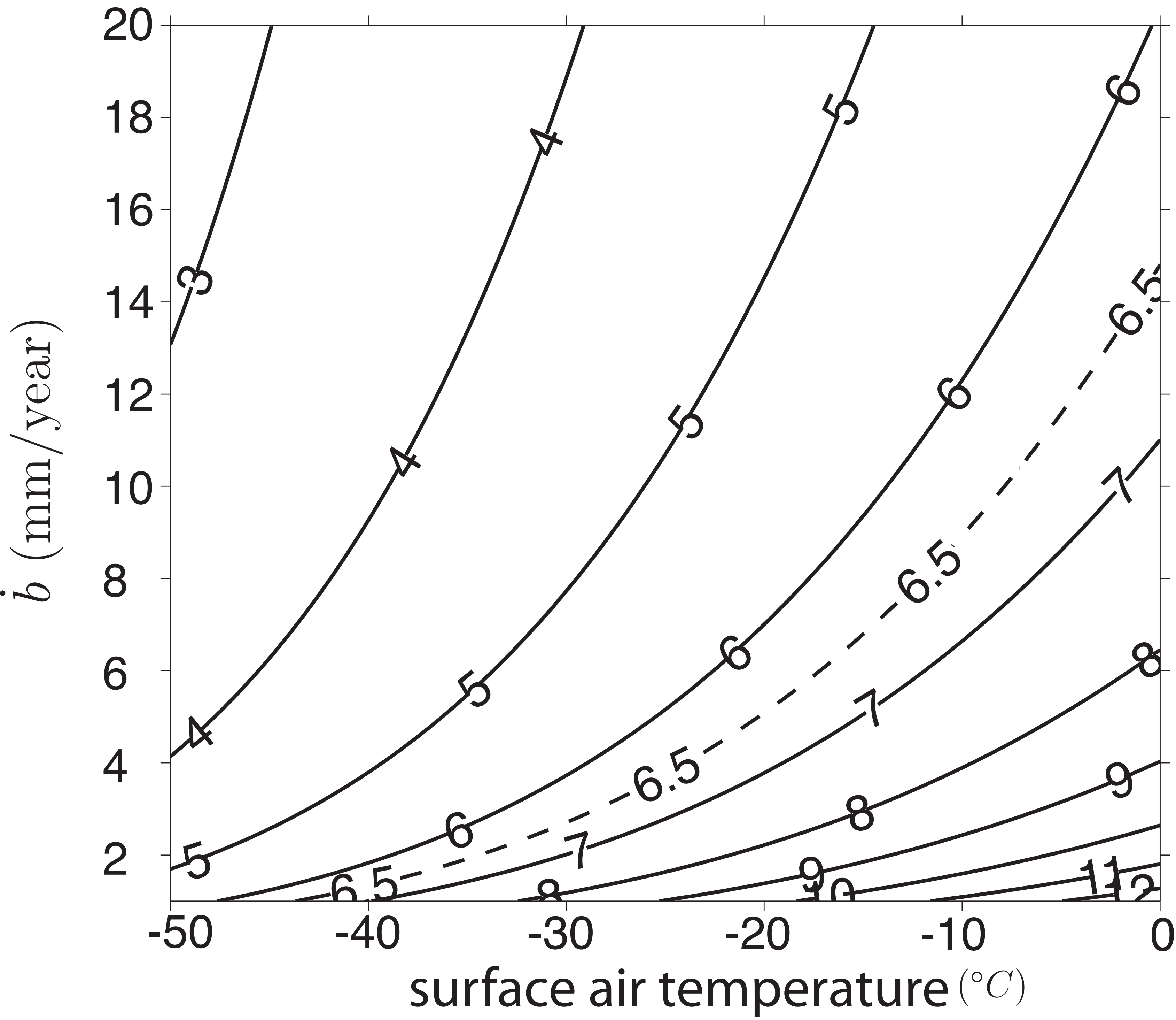
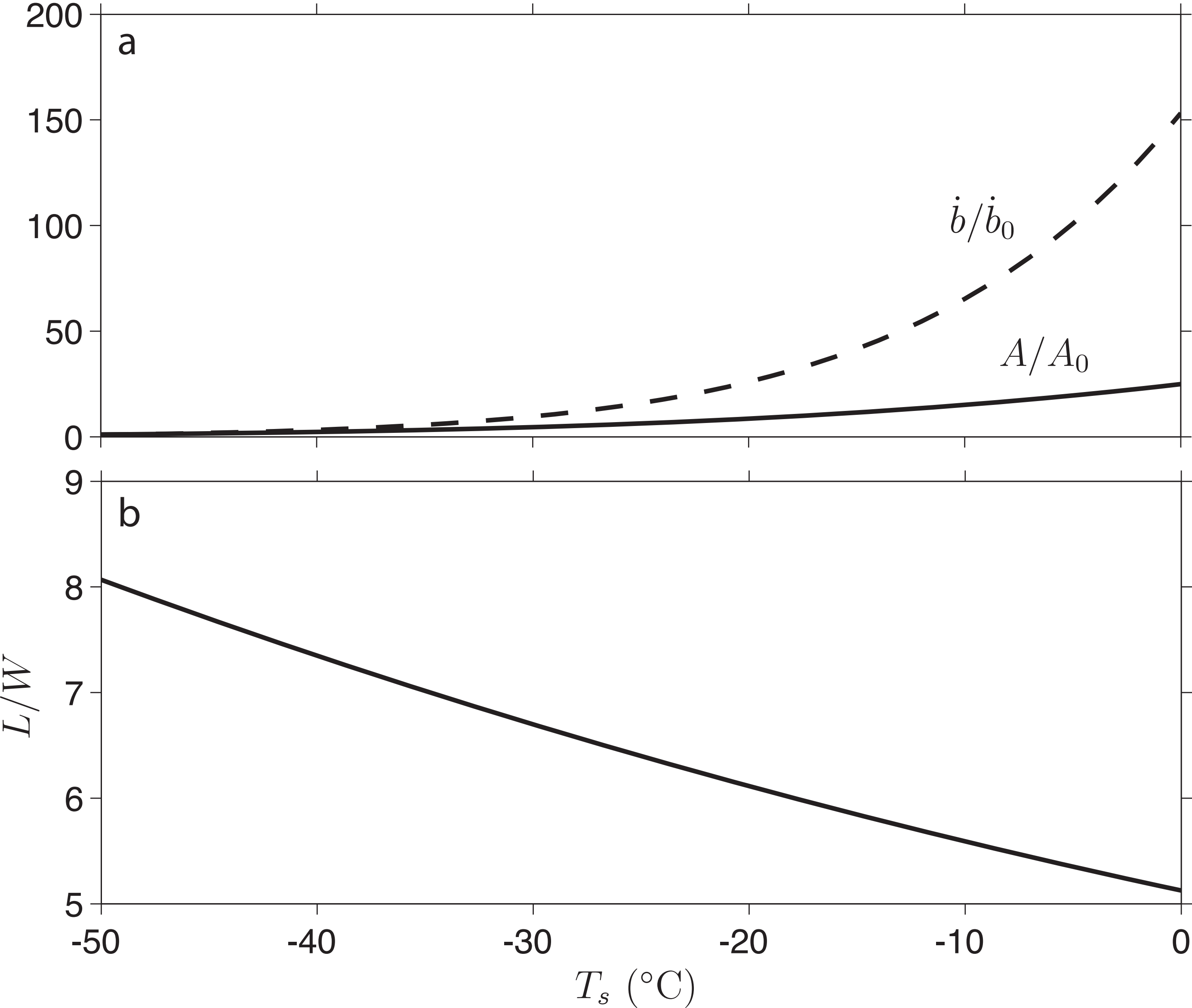
x̂
b
Q_in
0
distance (x)



20
18
16
14
12
10
8
6
4
2
$\dot{b}$ (mm/year)
-50
-40
-30
-20
-10
0
surface air temperature (°C)
3
4
5
6
7
8
9
10
11
12
-6.5



a
b
$\dot{b}/\dot{b}_0$
A/A_0
200
150
100
50
0
9
8
7
6
5
L/W
-50
-40
-30
-20
-10
0
T_s (°C)

Calculation of resistive drag stress

In order to calculate the resistive drag stress k along the sidewall of a sea glacier flowing through a narrow channel, a force-balance technique is used. Consider a vertical section of a sea glacier (Figure S-1) flowing through a narrow channel of uniform width W. The section has infinitesimal length Δx, where the thickness of the upstream and downstream ends are represented by H_U and H_D respectively. Pressure from ice at the upstream end P_U, acts to push the ice block downstream, while pressure from ice at the downstream end P_D, pressure from seawater P_{SW}, and resistive drag stress k from the sidewalls all act to resist the downstream motion of the ice block. We assume that the sea glacier is not accelerating, and that only pressure and drag forces act on the glacier, so that

$$F_U + F_D + F_{SW} + F_R = 0 \tag{S-1}$$

where F_U, F_D and, F_{SW} respectively represent forces due to pressure applied from ice at the upstream end, from ice at the downstream end, and from sea water at the base, and F_R represents the resistive drag force applied by the sidewalls. We assume that the stress deviator in the x-direction is zero, meaning that the compressive stress in the x-direction is due to ice pressure. This assumption leads to our uniform velocity, $\bar{u}$. *Sanderson*

[1979] found that the along-channel velocity is uniform for an ice shelf moving through a parallel-sided bay when he assumed a constant resistive drag stress k.

Pressure forces are calculated by integrating pressure over the appropriate face. Pressure at the upstream end of the section is linear with depth

$$P_U(z) = \Gamma\left(H_U\left(1 - \frac{\rho_i}{\rho_w}\right) - z\right) \tag{S-2}$$

where Γ is defined

$$\Gamma \equiv \rho_i g\left(1 - \frac{\rho_i}{\rho_w}\right) \tag{S-3}$$

Here, z is positive upward and sea level is at $z = 0$. The force applied at the upstream end of the thin cross-sectional element from pressure is calculated by integration over the upstream face

$$F_U = \int_{-H_U\left(\frac{\rho_i}{\rho_w}\right)}^{H_U\left(1-\frac{\rho_i}{\rho_w}\right)} \int_{-W/2}^{W/2} P_U(z)\,\mathrm{d}y\,\mathrm{d}z = \frac{W}{2}\rho_i g H_U^2 \tag{S-4}$$

Force at the downstream end is calculated similarly

$$F_D = -\frac{W}{2}\rho_i g H_D^2 \tag{S-5}$$

where a negative sign indicates force acting in the upstream direction.

Pressure of the sea water is linear with depth below sea level

$$P_{SW}(z) = \rho_w g(-z) \tag{S-6}$$

The component of force due to the pressure of seawater directed horizontally is calculated by integrating over the face from the base of the downstream end, to the base of the upstream end of the section.

$$F_{SW} = \int_{-H_U\left(\frac{\rho_i}{\rho_w}\right)}^{-H_D\left(\frac{\rho_i}{\rho_w}\right)} \int_{-W/2}^{W/2} P_{SW}(z)\,\mathrm{d}y\,\mathrm{d}z = -\frac{W}{2}\rho_i g \left(\frac{\rho_i}{\rho_w}\right)\left(H_U^2 - H_D^2\right) \tag{S-7}$$

where a negative sign indicates force acting in the upstream direction.

Substitution of Equations S-4, S-5, and S-7 into Equation S-1 yields

$$F_R = -\frac{W}{2}\Gamma[H_U^2 - H_D^2] \tag{S-8}$$

H_D can be described as a first-order Taylor expansion about H_0

$$H_D = H_U + \left.\frac{\mathrm{d}H}{\mathrm{d}x}\right|_{x_U} \Delta x \tag{S-9}$$

where the upstream face is at $x = x_U$

Substitution of Equation S-9 into Equation S-8 yields

$$F_R = -\frac{W}{2}\Gamma\left[-2H_U\frac{\mathrm{d}H}{\mathrm{d}x}\Delta x + \left(\frac{\mathrm{d}H}{\mathrm{d}x}\Delta x\right)^2\right]. \tag{S-10}$$

Since Δx is very small we can neglect terms containing $(\Delta x)^2$, and

$$F_R \approx W H_U \Gamma \frac{\mathrm{d}H}{\mathrm{d}x}\Delta x. \tag{S-11}$$

The resistive drag force along the two sidewalls is the integral of the resistive drag stress k over the sidewalls, where a negative sign indicates force acting in the upstream direction:

$$F_R = -2\int_x^{x+\Delta x}\int_0^{H(x)} k(x)\,\mathrm{d}z\,\mathrm{d}x = 2k(x)H(x)\Delta x. \tag{S-12}$$

We can now substitute Equation S-11 into Equation S-12 to calculate resistive drag stress:

$$k(x) = -\frac{W}{2}\Gamma\frac{\mathrm{d}H}{\mathrm{d}x}, \tag{S-13}$$

which turns out to be uniform in our analysis.

Calculation of effective softness and temperature

Our analytical solution (Equations 13 and 14) for the sea-glacier profile in an embayment requires isothermal ice. From Equations 3, 7, and 16, the isothermal ice flux through the cross-section at x is given by

$$Q_{iso}(x) = \frac{-A(x)}{2(n+2)}\left(\frac{W}{2}\Gamma\left|\frac{\mathrm{d}H}{\mathrm{d}x}\right|\right)^n W^2 H(x). \tag{S-14}$$

However, on Snowball Earth, the basal temperature would have been at the freezing point of seawater (approximately -2°C), while the upper surface was at the ambient air temperature T_s, so A then also had a strong depth dependence. When temperature varies with depth, the velocity profile across the channel at any depth z is still described by a form of Equation 2,

$$u(x,y,z) = \frac{W}{2}\frac{A(x,z)k(x,z)^n}{(n+2)}\left(1-\left|\frac{2y}{W}\right|^{n+1}\right). \tag{S-15}$$

With the assumption that vertical shear is negligible, i.e.

$$\frac{\partial u}{\partial z} = 0 \tag{S-16}$$

the cold upper ice acts as a stress guide, and wall drag $k(x,z)$ must be larger in the cold ice where softness $A(x,z)$ is smaller, so that

$$A(x,z)k(x)^n = C \tag{S-17}$$

where C is a constant that can be determined. The softness $A(x,z)$ is known from the temperature profile, and the depth-averaged value of $k(x,z)$ must equal the value of $k(x)$ in Equation 7 in the isothermal case for the same sea-glacier geometry, in order to balance the forces in Equation S-1. As a result,

$$C = \left[\frac{-\frac{W}{2}\Gamma\left|\frac{\mathrm{d}H}{\mathrm{d}x}\right|}{\frac{1}{H(x)}\int_0^{H(x)} A(x,\zeta)^{-1/n}\mathrm{d}\zeta}\right]^n \tag{S-18}$$

and the ice flux can be written as

$$Q(x) = WH(x)\bar{u}(x) = \frac{W^2}{2(n+2)}H(x)C = \frac{-W^2H(x)}{2(n+2)}\left(\frac{W}{2}\Gamma\left|\frac{\mathrm{d}H}{\mathrm{d}x}\right|\right)^n\left(\frac{1}{H(x)}\int_0^{H(x)} A(x,\zeta)^{-1/n}\mathrm{d}\zeta\right)^{-n}. \tag{S-19}$$

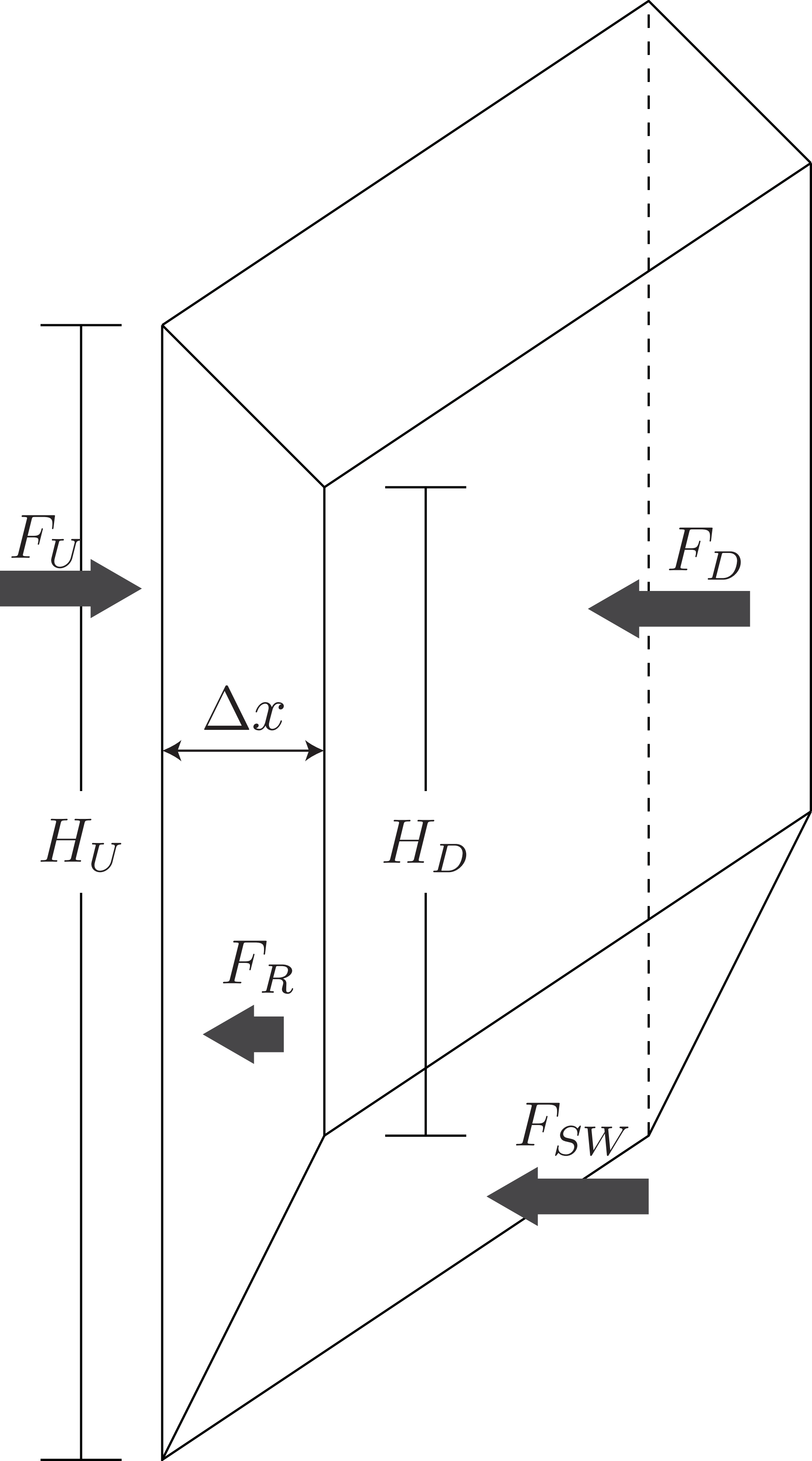
The depth integral can be evaluated numerically from the known temperature profile $T(x,z)$ and Equation 1. We can then define an effective ice softness $A_{eff}(x)$for the cross-section as

$$A_{eff}(x) = \left(\frac{1}{H(x)}\int_0^{H(x)} A(x,\zeta)^{-1/n}\mathrm{d}\zeta\right)^{-n}. \tag{S-20}$$

With the substitution Equation S-20, Equation S-19 takes the same form as Equation S-14, so ice flux through any cross-section in a non isothermal sea glacier can be calculated correctly with the isothermal Equations 15 and 16, by replacing $A(x)$ with $A_{eff}(x)$ from Equation S-20.

Furthermore, when the temperature profile is linear with depth, and the surface temperature T_s is spatially uniform, A_{eff} is independent of ice thickness $H(x)$, and so is also independent of x. A uniform effective temperature T_{eff} for the entire sea glacier can then be derived from A_{eff} through Equation 1.

Figure S-1. A section of a sea glacier penetrating a channel experience forces acting on the upstream face from the pressure of ice F_U, on the downstream face from the pressure of ice F_D, on the basal face from the pressure of seawater F_{SW}, and on the sidewalls from drag resistance F_R.


F_U
F_D
Δx
H_U
H_D
F_R
F_{SW}