A Two-Box Model of Cloud-Weighted Sea-Surface Temperature:

The Semi-Automatic Negative Correlation with Mean Cloud Fraction

Dennis L. Hartmann and Marc L. Michelsen

Supplementary material for BAMS response to Lindzen et al.

April 2002.

The system of interest consists of two equal areas with SSTs of T_c and T_w and cloud area fractions of C_c and C_w. In this case the cloud-weighted SST (CWT) and average cloud cover, A, are given by:

$$CWT = \frac{C_w T_w + C_c T_c}{2A} \quad \text{and} \quad A = \frac{1}{2}(C_w + C_c) \tag{1}$$

We will assume that the SST remains fixed with time, so that the variations in *CWT* arise solely from variations in cloud coverage. Defining $\Delta T = T_w - T_c$, and noting that $C_w = 2A - C_c$, we can write:

$$CWT = T_w - \frac{C_c}{C_c + C_w}\Delta T = T_c + \frac{C_w}{C_c + C_w}\Delta T \tag{2}$$

Intuitively, it seems that if the variance of the cloud cover in the cold region is greater than the variance in the warm region, then the ratio $\frac{C_c}{C_c + C_w}$ will be positively correlated with A and the ratio $\frac{C_w}{C_c + C_w}$ will be negatively correlated with *A*, so that *CWT* will perforce be negatively correlated with *A*. The following mathematical analysis of this intuition was provided to us by our colleague Christopher S. Bretherton.

The correlation between the mean cloud cover A and the factor $F = \frac{C_c}{C_c + C_w}$ is proportional to the covariance of F with 2A, $Cov\{F, 2A\}$. Assume that the cloudiness values in the two regions are random time series with known means and standard deviations.

$$
\begin{aligned}
C_c &= \overline{C_c} + \sigma_c n_c \\
C_w &= \overline{C_w} + \sigma_w n_w
\end{aligned} \tag{4}
$$

where n_c *and* n_w are random timeseries with zero mean and unit variance. Further define $r = Cov\{n_c, n_w\}$, the correlation between the random time series that make up the time-varying part of the cloudiness timeseries for the two regions. It will also be necessary to assume that

$$
\frac{\sigma_c}{\overline{C_c}} << 1 \ \ and \ \ \frac{\sigma_w}{\overline{C_c} + \overline{C_w}} << 1 \tag{5}
$$

Then,

$$
2A = C_c + C_w = \overline{C_c} + \overline{C_w} + \sigma_c n_c + \sigma_w n_w \tag{6}
$$

and

$$
\begin{aligned}
F = \frac{C_c}{C_c + C_w} &= \frac{\overline{C_c}}{\overline{C_c} + \overline{C_w}} \left(\frac{1 + \frac{\sigma_c n_c}{\overline{C_c}}}{1 + \frac{\sigma_c n_c + \sigma_w n_w}{\overline{C_c} + \overline{C_w}}} \right) \\
&\approx \frac{\overline{C_c}}{\overline{C_c} + \overline{C_w}} \left(1 + \frac{\sigma_c n_c}{\overline{C_c}} - \frac{\sigma_c n_c + \sigma_w n_w}{\overline{C_c} + \overline{C_w}} \right)
\end{aligned} \tag{7}
$$

so that,

$$
\begin{aligned}
Cov\{F, 2A\} &\approx \frac{\overline{C_c}}{\overline{C_c} + \overline{C_w}} Cov\left\{ \frac{\sigma_c n_c}{\overline{C_c}} - \frac{\sigma_c n_c + \sigma_w n_w}{\overline{C_c} + \overline{C_w}}, \ \sigma_c n_c + \sigma_w n_w \right\} \\
&= \frac{1}{\left(\overline{C_c} + \overline{C_w}\right)^2} \left(\sigma_c{}^2 \, \overline{C_w} - \sigma_w{}^2 \, \overline{C_c} + \sigma_c \sigma_w \, r \left(\overline{C_w} - \overline{C_c} \right) \right)
\end{aligned} \tag{8}
$$

If $r = 0$, then $Cov\{F, 2A\} > 0$ under the condition that,

$$\frac{{\sigma_c}^2}{{\sigma_w}^2} > \frac{\overline{C_c}}{\overline{C_w}} \tag{9}$$

The same condition (9) will guarantee that,

$$Cov\left\{\frac{C_w}{C_c + C_w}, C_c + C_w\right\} < 0 \tag{10}$$

These derivations suggest that a negative correlation between cloud-weighted SST and cloud amount should be expected if the cloud amount in the cold region has relatively low mean cloudiness and/or high variability compared to the warm area.