Amath-Math 586/Atm S 581 Due Friday, 12 May 2005

Homework 4 (Please do by yourself without consulting with anyone)

A classic nonlinear hyperbolic equation that describes traffic flow along a length of road $0 < X < L$ with no entrance or exit ramps (e.g. the SR520 bridge) is:

$$Q_T + F_X = 0, \qquad (*)$$

where $Q(X, T)$ is the number of cars per unit length of road and $F(X, T)$ describes the traffic flux (number of cars per unit time crossing position X at time T), which is Q multiplied by the car speed $U(Q)$. A simple model is to take $U(Q) = U_0(1 - Q/Q_0)$, where U_0 is the speed limit and Q_0 is a car density so high that the traffic gets stopped. Note that (*) can also be written in the'advective form'

$$Q_T + C(Q)Q_X = 0, \quad C(Q) = dF/dQ = U_0(1 - 2Q/Q_0). \qquad (*')$$

Suppose that at $t = 0$ there are no cars on the road: $q(x,0) = 0$ for $0 < x < L$ at $x = 0$, and that for $t > 0$, the traffic density $Q(0, T)$ is known. This 'signalling problem' is a nonlinear analogue of HW3 problem 2.

1. Find an appropriate nondimensionalization $q = AQ$, $x = BX$, $t = CT$ that transforms (*') into an identical nondimensional system with $Q_0 = U_0 = L = 1$:

   $$q_t + c(q)q_x = 0, \quad c(q) = df/dq = 1 - 2q \quad \text{(nondim. flux } f(q) = q(1\text{-}q)) \qquad (*'')$$

   $$q(x, 0) = 0 \text{ for } 0 < x < 1, \quad q(0, t) \text{ known for } t > 0.$$

2. Suppose $q(0, t) = 0.12\sin^2(\pi t)$ for all $t > 0$. Using the method of characteristics, calculate and plot the analytical solution $q_{ex}(1, t)$ of (*'') at $x = 1$ for $0 < t < 3$.
3. Modify your Matlab script from HW3 problem 2 to solve (*'') using the leapfrog centered-in-space method with midpoint-RK2 starting step, a one-sided difference at $x = 1$, and an Asselin filter with $\gamma = 0.05$, using $\Delta x = 0.05$ and $\Delta t = 0.025$ (much less than the CFL limit of 0.05, but used here to help suppress high-frequency oscillations). Overplot $q(1, t)$ for $0 < t < 3$ on the exact solution and calculate its max-norm error. Do you see spurious oscillations?
4. Consider a piecewise linear finite volume method for (*'') with a downwind slope σ_j. The appropriate generalization of this method to a conservation law with nonconstant wave speed $c(q) = df/dq > 0$ (as in our problem) is to use a numerical flux

   $$F_{j+1/2}^n = f(q_j^n) + 0.5c_{j+1/2}^n \sigma_j^n \Delta x \left(1 - \mu_{j+1/2}^n\right)$$

   $$c_{j+1/2}^n = \frac{f(q_{j+1}^n) - f(q_j^n)}{q_{j+1}^n - q_j^n}, \quad \mu_{j+1/2}^n = c_{j+1/2}^n \Delta t / \Delta x,$$

   so $$q_j^{n+1} = q_j^n - (\Delta t/\Delta x)\left(F_{j+1/2}^n - F_{j-1/2}^n\right) \quad \text{at } x_j = j\Delta x,\ j = 1,\ldots, N, \text{ where } x_0=0,\ x_N = 1.$$

   Implement the downwind slope (Lax-Wendroff) method $\sigma_j^n = (q_{j+1}^n - q_j^n)/\Delta x$, and the MC slope-limiter, assuming left and right ghost cells with $q_{-1}^n = q_0^n$ and $q_{N+1}^n = q_N^n$. Overplot the calculated $q(1, t)$ for both methods for $0 < t < 3$ on your results for problem 3, using a grid spacing of $\Delta x = 0.05$ and a timestep $\Delta t = 0.045$. Are spurious oscillations evident? Compare their max-norm errors to problem 3. Which method gives the best solution?