Amath-Math 586/Atm S 581 Due Wednesday, 3 May 2005

Homework 3

1. Consider the system of equations describing two chemical species *a* and *b* of concentrations $A(t)$ and $B(t)$, respectively, for which species *a* has a source $1 - \cos(t)$ but rapidly decays into *b*, and *b* decays much more slowly:

$$dA/dt = -\lambda A + 1 - \cos(t), \quad \lambda >> 1$$

$$dB/dt = \lambda A - B, \qquad A(0) = B(0) = 0. \quad (*)$$

   (a) By considering the dominant balance in the equation for *a* (please don't go to the trouble of an exact solution of the equation valid for finite λ), show that

   $B(t) = 1 - 0.5e^{-t} - 0.5[\cos(t) + \sin(t)] + O(\lambda^{-1})$

   (b) Use a backward Euler method to solve the IVP (*) with $\lambda = 1000$ for $0 < t < T = \pi$. In particular, find the numerical solution $B_{\Delta t}(T)$ for $\Delta t = T/N$, where $N = 2^{-p}$, $p = 2,\ldots,7$. Since the exact solution is a bit messy, measure error for each Δt using the solution increment $\varepsilon(\Delta t) = |B_{\Delta t}(T) - B_{2\Delta t}(T)|$. Log-log plot ε vs. timestep Δt for $p = 3,\ldots,7$. Does ε converge with Δt at the expected rate?

   (c) Now use the BDF2 method with the same range of Δt, with a backward Euler starting step. Overplot the BDF2 solution increments. Are they smaller than with backward Euler? Do they converge with Δt at the expected rate?

2. Consider the following IBVP, describing advection of a tracer $u(x, t)$ introduced into water flowing into one end of a pipe. The pipe narrows with increasing *x*, causing a corresponding increase in the flow speed that carries the tracer along:

$$u_t + (1+9x)u_x = 0, \quad 0 < x < 1, 0 < t < 0.5$$

$$u(x,0) = 0, u(0, t) = 1$$

   (a) What is the exact solution $u_e(1,t)$? Why is no boundary condition required for $x = 1$?

   (b) Consider the leapfrog method with centered space differencing for the derivative, using a midpoint second-order Runge-Kutta starting timestep. If we use a small uniform grid spacing Δx, what is the CFL stability limit on Δt?

   (c) At the right boundary $x = 1$, we lack the information to calculate a centered difference. Give a second-order accurate one-sided approximation to u_x that we can use there.

   (d) Implement the above method using $\Delta x = 0.05$ and Δt given by the stability limit found in (b). Use Asselin filtering with $\gamma = 0.05$. Plot a comparison of the numerical and the exact solutions at $x = 1$ for $0 < t < 0.5$. Are the dominant errors diffusive or dispersive? Is this as you expected? – Explain. What happens if without Asselin filtering (i.e. $\gamma = 0$)?