Amath-Math 586/Atm S 581 Spring 2005

Homework Set 2 Solutions

1. Taylor-expanding the component terms in the LTE in x and t about (x_j, t_n), and keeping terms through second order in Δt and Δx, we obtain:

$$\frac{\psi_j^{n+1}-\psi_j^n}{\Delta t}=\psi_t+\frac{\Delta t}{2}\psi_{tt}+\frac{\Delta t^2}{6}\psi_{ttt}+O(\Delta t^3)$$

$$\frac{\psi_{j+1}^{n}-\psi_{j-1}^n}{2\Delta x}=\psi_x+\frac{\Delta x^2}{6}\psi_{xxx}+O(\Delta x^4)$$

$$\frac{\psi_{j+1}^{n+1}-\psi_{j-1}^{n+1}}{2\Delta x}=\left(\psi+\Delta t\psi_t+\frac{\Delta t^2}{2}\psi_{tt}\right)_x+\frac{\Delta x^2}{6}\psi_{xxx}+O(\Delta x^4,\Delta x^2\Delta t,\Delta t^3)$$

so

$$\text{LTE}=\frac{\psi_j^{n+1}-\psi_j^n}{\Delta t}+\frac{c}{2\Delta x}\left(\frac{\psi_{j+1}^{n+1}+\psi_{j+1}^n}{2}-\frac{\psi_{j-1}^{n+1}+\psi_{j-1}^n}{2}\right)$$

$$=\psi_t+\frac{\Delta t}{2}\psi_{tt}+\frac{\Delta t^2}{6}\psi_{ttt}+c\left(\psi+\frac{\Delta t}{2}\psi_t+\frac{\Delta t^2}{4}\psi_{tt}\right)_x+c\frac{\Delta x^2}{6}\psi_{xxx}+O(\Delta x^4,\Delta x^2\Delta t,\Delta t^3)$$

Since ψ is the exact solution of the advection equation, $\psi_t=-c\psi_x$, $\psi_{tt}=-c\psi_{tx}$, and $\psi_{ttt}=-c\psi_{txx}=-c^3\psi_{xxx}$. These relations can be used to simplify the LTE to

$$\text{LTE}=c\psi_{xxx}\left(-\frac{c^2\Delta t^2}{6}+\frac{c^2\Delta t^2}{4}+\frac{\Delta x^2}{6}\right)+O(\Delta x^4,\Delta x^2\Delta t,\Delta t^3)\,,$$

$$=c\psi_{xxx}\frac{\Delta x^2}{12}\left(\mu^2+2\right)+O(\Delta x^4,\Delta x^2\Delta t,\Delta t^3)\,,$$

where $\mu = c\Delta t/\Delta x$. This shows the method is second-order accurate in x, t.

2. Using a von Neumann analysis, show that the numerical dispersion relation is

$$\tan(\omega\Delta t/2)=\frac{c\Delta t}{2\Delta x}\sin(k\Delta x)$$

Since the FDA is constant-coefficient, we look for sinusoidal eigenmodes $\phi_j^n=\exp(ikx_j-i\omega t_n)$. Factoring out a term $\mathrm{P}=\exp(ikx_j-i\omega t_{n+1/2})$ from all terms in the FDA,

$$\frac{\psi_j^{n+1}-\psi_j^n}{\Delta t}=\frac{e^{i\omega\Delta t/2}-e^{-i\omega\Delta t/2}}{\Delta t}P=\frac{2i\sin(\omega\Delta t/2)}{\Delta t}P$$

$$\frac{c}{2}\left(\frac{\psi_{j+1}^{n+1}-\psi_{j-1}^{n+1}}{2\Delta x}+\frac{\psi_{j+1}^{n}-\psi_{j-1}^{n}}{2\Delta x}\right)=\frac{c}{2}\left(\frac{e^{ik\Delta x}-e^{-ik\Delta x}}{2\Delta x}\right)\left(e^{i\omega\Delta t/2}+e^{-i\omega\Delta t/2}\right)P=\frac{ic\sin(k\Delta x)}{\Delta x}\frac{\cos(\omega\Delta t/2)}{\Delta t}P$$

so the FDA becomes (cancelling a factor *P* from both terms above),

$$0=\frac{-2i\sin(\omega\Delta t/2)}{\Delta t}+\frac{ic\sin(k\Delta x)}{\Delta x}\frac{\cos(\omega\Delta t/2)}{\Delta t}$$

$$\tan(\omega\Delta t/2)=\mu\sin(k\Delta x)/2$$

3. For any real *k*, the above dispersion relation has a real root

$$\omega\Delta t/2=\tan^{-1}\left[\mu\sin(k\Delta x)/2\right] \qquad (3.1)$$

such that $|\omega\Delta t| < \pi$. This implies that the amplification factor A = exp(-i $\omega\Delta t$) has unit amplitude (so the method is neutral). To show the method is decelerating, expand (3.1) in a Taylor series for small $k\Delta x$:

$$\omega\Delta t/2=\tan^{-1}\left[\frac{\mu}{2}(k\Delta x-\frac{\{k\Delta x\}^3}{6}...)\right]$$

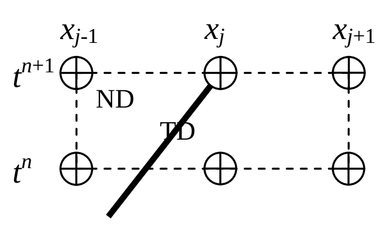
$$=\left[\frac{\mu}{2}(k\Delta x-\frac{\{k\Delta x\}^3}{6}...)\right]-\frac{1}{3}\left[\frac{\mu}{2}(k\Delta x-\frac{\{k\Delta x\}^3}{6}...)\right]^3...$$

$$=\frac{\mu k\Delta x}{2}-\frac{\mu\{k\Delta x\}^3}{12}-\frac{\mu^3\{k\Delta x\}^3}{24}+O(\{k\Delta x\}^5)$$

$$\omega=ck\left[1-\frac{\{k\Delta x\}^2}{12}(2+\mu^2)+O(\{k\Delta x\}^4)\right] \qquad (3.2)$$

Since $\omega/\omega_{ex} = \omega/ck < 1$, the method is decelerating. Notice this result has a form reminiscent of the LTE analysis, which is no coincidence, since the LTE analysis would imply a modified equation $\psi_t + c\psi_x = LTE = c\psi_{xxx}\Delta x^2\left(\mu^2+2\right)/12+....$, which must have (and does have) a dispersion relation equal to the leading order numerical dispersion (3.2).

4. The stencil for the method is shown at right. For any Courant number, the interior ND of its convex hull (the numerical domain of dependence) clearly includes a true domain of dependence TD consisting of a finite section of the line $x - x_j = c(t - t_{n+1})$ that has $t < t_{n+1}$, so there is no CFL stability limit. This is due to the method being implicit, involving ϕ_{j-1}^{n+1} and ϕ_{j+1}^{n+1} in the calculation of ϕ_j^{n+1} .



5. From problem 3 (or 1), the leading order error in the dispersion relation for small $k\Delta x$ is $\delta\omega=-ck\left(k\Delta x\right)^2(2+\mu^2)/12$. The component proportional to μ^2 arises from the time

differencing error and the other component from the space differencing error. Hence the space and time differencing errors are equal if $\mu^2 = 2$.

6. If $\Delta t = LT/(M\Delta x)$, the Courant number $\mu = c\Delta t/\Delta x = Q/\Delta x^2$, where $Q = cLT/M$ is a fixed parameter. The error in both the numerical dispersion relation and the LTE are both proportional to $\varepsilon = \Delta x^2(2+\mu^2) = 2\Delta x^2 + Q^2\Delta x^{-2}$. Minimizing ε w.r.t. Δx,

$$0 = \partial\varepsilon/\partial\Delta x = 4\Delta x - 2Q^2\Delta x^{-3},$$

from which we conclude $\Delta x^4 = Q^2/2$. This is the value of Δx for which the two terms in ε are equal, i.e. for which $\mu^2 = 2$. Thus, choosing this Courant number should produce the most accurate solution for a given number of computations.

6. If the grid spacing is $\Delta x = L/N$, we choose gridpoints $x_j = (j-1)/N$, $j = 1,\ldots,N+2$. These include an image point on each side in addition to the N interior gridpoints $j = 2,\ldots,N+1$ at which we advance the numerical solution using the FDA. At each time level n, we specify the image point values $\phi_1^n = \phi_{N+1}^n$ and $\phi_{N+2}^n = \phi_2^n$. We also apply the same BCs at time level $n+1$ in the implicit solver.

7. If the wave speed c depended on the solution: $c_j^n = F(\phi_j^n)$, then the trapezoidal-centered FDA

$$0 = \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + \frac{c(\phi_j^{n+1})}{2\Delta x}\left(\frac{\phi_{j+1}^{n+1} + \phi_{j+1}^n}{2} - \frac{\phi_{j-1}^{n+1} + \phi_{j-1}^n}{2}\right).$$

would be a nonlinear system of nearest-neighbor-coupled equations in the unknowns $\{\phi_j^{n+1}\}$, which one would have to solve via a method such as Newton iteration. Depending on how much iteration were required, this would reduce the computational efficiency of the method enough that it might not be competitive with explicit methods, even though those methods would be CFL-limited to running with a smaller Courant number.

8. The Matlab script hw2p8.m makes the plot below of the solution error. Note that we use the grid 2-norm, which is the conventional 2-norm times $\Delta x^{1/2}$, to allow proper comparison of the error across different grid spacings. The method is 2nd order accurate, as expected.