Amath-Math 586/Atm S 581 Due Wednesday, 20 April, 2005

Homework Set 2

We wish to solve the advection equation $\psi_t + c\psi_x = 0$ on the domain $0 < x < L$, with the periodic boundary condition $\psi(0) = \psi(L)$. We use trapezoidal time differencing and centered space differencing on a uniform grid:

$$0 = \delta_t \phi_j^{n+1/2} + \delta_{2x}\phi_j^{n+1/2} = \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + \frac{c}{2\Delta x}\left(\frac{\phi_{j+1}^{n+1} + \phi_{j+1}^n}{2} - \frac{\phi_{j-1}^{n+1} + \phi_{j-1}^n}{2}\right)$$

1. Using a local truncation error analysis, deduce the order of accuracy of this method in *x* and *t*.
2. Using a von Neumann analysis, show that the numerical dispersion relation is
$$\tan(\omega\Delta t / 2) = \frac{c\Delta t}{2\Delta x}\sin(k\Delta x)$$
3. From this dispersion relation, show that this method is neutral and decelerating for all Courant numbers.
4. How is this stability is consistent with the CFL stability condition? (Hint: method is *implicit*)
5. Calculate the leading order errors in the dispersion relation in $k\Delta x$ and $\omega\Delta t$ for waves well-resolved by our space time grid. Show that the space and time differencing errors are equal if the Courant number $\mu = c\Delta t/\Delta x = 2$.
6. Supposing our computer resources limit us to applying the method for a fixed maximum number *M* of gridpoint-timesteps. Suppose we want to integrate out to a given time *T*. We need to choose an optimal timestep Δt, and reason as follows. For any given Δt, we can use $M\Delta t/T$ gridpoints, or a grid spacing $\Delta x = LT/(M\Delta t)$. Given the result in problem (5), show if our goal is to minimize the overall numerical error that we make in the dispersion relation for well-resolved waves, we should choose Δt such that $\mu = 2$.
6. What is a suitable choice of image points and numerical BC's to enforce periodicity in *x*?
7. Suppose that the wave speed *c* were not constant, but depended on the solution: $c_j^n = F(\phi_j^n)$. What would be the difficulty in implementing this method?
8. Taking $c = L = 1$, write a Matlab script using the Matlab sparse matrix software (type `help spdiags` in Matlab if you are unfamiliar with this) to implement this method for $\psi(x, 0) = \sin(2\pi x)$. Use a Courant number of 1 and grid spacing $\Delta x = 0.2, 0.1, 0.05$, and 0.025, calculate the 2-norm solution error at t = 1. Log-log plot the solution error vs. Δx and check whether if the method converges at the expected rate.