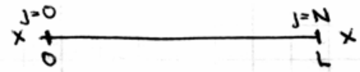
Amath-Math 586/Atm. Sci. 581 Spring 2005

HW1 Solutions

(a) Define points ϕ_{-1}^n and ϕ_{N+1}^n. Then BC is



$$\phi_{-1}^n = \phi_1^n$$
$$\phi_{N+1}^n = \phi_{N-1}^n$$

(b) Plug exact solution into FTCS method

$$T = \delta_t^F \psi(x_j,t_n) - a\,\delta_x^2 \psi(x_j,t_n)$$

Using Taylor expansion about (x_j, t_n),

$$\delta_t^F \psi(x_j,t_n) = \frac{\psi(x_j, t_n+\Delta t) - \psi(x_j,t_n)}{\Delta t} = \psi_t + \frac{\Delta t}{2}\psi_{tt} \cdots$$

(all quantities evaluated at x_j, t_n)

$$\delta_x^2 \psi(x_j,t_n) = \frac{\psi(x_j+\Delta x, t_n) - 2\psi(x_j,t_n) + \psi(x_j-\Delta x, t_n)}{(\Delta x)^2} = \psi_{xx} + \frac{(\Delta x)^2}{12}\psi_{xxxx}\cdots$$

$$\Rightarrow T = \underbrace{\cancel{\psi_t} + \frac{\Delta t}{2}\psi_{tt} - a\Big(\cancel{\psi_{xx}}}_{\text{cancel since } \psi_t = a\psi_{xx}} + \frac{(\Delta x)^2}{12}\psi_{xxxx}\cdots\Big)$$

$$= \frac{\Delta t}{2}\psi_{tt} - \frac{a(\Delta x)^2}{12}\psi_{xxxx} + \text{h.o.t.}$$

Since $T = O(\Delta t, \Delta x^2)$, method is 1st order accurate in t, 2nd order in x.

(c) For the exact PDE and BC's,

$$\frac{\partial}{\partial t}\int_0^L \psi\, dx = \int_0^L \psi_t\, dx = \int_0^L a\psi_{xx}\, dx = a\big[\psi_x(L,t) - \psi_x(0,t)\big] = 0.$$

$$\frac{\partial}{\partial t}\int_0^L \psi^2 dx = 2\int_0^L \psi\psi_t\, dx = 2a\int_0^L \psi\psi_{xx}\, dx$$
$$= 2a\left\{\cancel{\big[\psi\psi_x\big]_0^L}^{\,0} - \int_0^L \psi_x^2\, dx\right\}$$
$$\leq 0.$$

(d) For the FTCS method, we can show the following discrete analogue to $\frac{\partial}{\partial t}\int_0^L \psi\, dx = 0$:

$$\delta_t^F \underbrace{\left\{\sum_{j=0}^{N} w_j \phi_j^n\right\}}_{I^n} = 0 \qquad \text{where } w_j = \begin{cases} \tfrac{1}{2} & j=0,N \\ 1 & 1 \leq j \leq N-1 \end{cases}$$

gives trapezoidal weighting to grid points

i.e. I^n remains constant with n. To see this, note

$$I^n = \sum_{j=0}^{N} w_j\, \delta_t^F \phi_j^n = a\sum_{j=0}^{N} w_j \left\{\frac{(\phi_{j+1}^n - \phi_j^n) - (\phi_j^n - \phi_{j-1}^n)}{\Delta x^2}\right\}$$

For $j=1,\dots,N-1$ this series telescopes, so

$$I^n = \frac{a}{\Delta x^2}\left\{\frac{1}{2}\left[(\phi_1^n-\phi_0^n)-(\phi_0^n-\phi_{-1}^n)\right]+\left[(\phi_N^n-\phi_{N-1}^n)-(\phi_1^n-\phi_0^n)\right] + \frac{1}{2}\left[(\phi_{N+1}^n-\phi_N^n)-(\phi_N^n-\phi_{N-1}^n)\right]\right\}$$

$$= \frac{a}{\Delta x^2}\left\{\frac{1}{2}\left[\phi_1^n-\phi_{-1}^n\right]+\frac{1}{2}\left[\phi_{N+1}^n-\phi_{N-1}^n\right]\right\}$$

$= 0$ by discrete BC's.

(e) Apply von Neumann analysis to FTCS by letting $\phi_j^n = e^{ikx_j}$,

$\phi_j^{n+1} = A_k e^{ikx_j}$:

$$\delta_t^F \phi_j^n = \frac{1}{\Delta t}(A_k-1)\, e^{ikx_j}$$

$$\delta_x^2 \phi_j^n = \frac{1}{\Delta x^2}\left(e^{ik\Delta x}-2+e^{-ik\Delta x}\right)\cdot e^{ikx_j}$$

$$\text{FTCS} \Rightarrow 0 = \frac{1}{\Delta t}(A_k-1) - \frac{a}{\Delta x^2}\left(2\cos k\Delta x - 2\right)$$

$$A_k = 1 - \frac{a\Delta t}{\Delta x^2}\cdot 2(1-\cos k\Delta x)$$

If we define $\nu = \frac{a\Delta t}{\Delta x^2}$, and take $\Delta x \to 0$ with fixed ν,

$$A_k = 1 - 2\nu(1-\cos k\Delta x)$$

$$= 1 - 2\nu\left(1-\left[1-\frac{(k\Delta x)^2}{2}+\frac{(k\Delta x)^4}{24}\cdots\right]\right)$$

$$= 1+\nu\cdot\left[-(k\Delta x)^2+\frac{(k\Delta x)^4}{12}\cdots\right]$$

The exact $A_{ex}(k)$ is the solution $\psi(x,\Delta t) = A_{ex}(k)e^{ikx}$

$$\psi_t = a\psi_{xx}$$

$$\psi(x,0) = e^{ikx}$$

$$\Rightarrow \psi(x,t) = e^{ikx-ak^2t}, \quad A_{ex}(k) = e^{-ak^2\Delta t} = e^{-\nu(k\Delta x)^2}$$

As $k\Delta x \to 0$ with ν fixed

$$\Rightarrow A_{ex}(k) = 1 - \nu(k\Delta x)^2 + \frac{\nu^2}{2}(k\Delta x)^4\dots$$

so

$$A_k - A_{ex}(k) = O(\nu^2\Delta x^4) = O(\Delta t^2)$$

(as we'd expect for a method 1st order accurate in t.)

Now, to look at stability, note that

$$1 \geq A_k \geq 1 - 4\nu$$

since $\cos k\Delta x$ achieves a maximum at $k=0$ of 1 and a minimum at $k\Delta x = \pi$ of -1. Thus,

$$\max_k |A_k| = \begin{cases} 1 & \text{for } \nu \leq \frac{1}{2} \\ 4\nu - 1 & \text{for } \nu > \frac{1}{2}. \end{cases}$$

Thus method is stable for $\nu \leq \frac{1}{2}$.

(f) For $\nu > \frac{1}{2}$, the $2\Delta x$ wave with $k = \frac{\pi}{\Delta x}$ amplifies by a factor $1 - 4\nu$ per timestep, so that if

$$\phi_j^0 = e^{ikx_j}$$

after $T/\Delta t$ timesteps

$$\phi_j^{T/\Delta t} = (1 - 4\nu)^{T/\Delta t}$$

As $\Delta t \to 0$, $\dfrac{|\phi_j^{T/\Delta t}|}{|\phi_j^0|} \to \infty$, so the method is unstable.

(g) The plot below was generated with the Matlab script hw1g.m posted on class web page. It demonstrates the expected $O(N^{-2})$ error convergence using a log-log plot of error.