Amath-Math 586/Atm S 581 Due Monday, April 11, 2005

Homework Set 1

Consider the initial-boundary value problem

$$\psi_t = a\psi_{xx}, \quad 0 < x < L,\ t > 0, \tag{1}$$

$$\psi(x, 0) = f(x),$$

$$\psi_x(0, t) = \psi_x(L, t) = 0.$$

In the *forward-in-time, centered-in-space* (FTCS) method, the PDE (1) is discretized as

$$\delta_t^F \phi_j^n = a\delta_x^2 \phi_j^n, \qquad x_j = j\Delta x, \qquad j = 0, ..., N, \qquad \Delta x = L/N. \tag{2}$$

(a) Using an image point method, specify appropriate boundary conditions for (2) at $j = 0, N$.

(b) Analyze the local truncation error and order of accuracy in x, t of the FTCS method.

(c) Show that for the exact PDE and initial/boundary conditions,

$$\frac{\partial}{\partial t}\int_0^L \psi\, dx = 0 \tag{c.1}$$

$$\frac{\partial}{\partial t}\int_0^L \psi^2 dx < 0 \tag{c.2}$$

(d) Show that the FTCS method preserves an appropriate discrete analogue to (c.1) with the BC's specified in (a). Hint: Consider a trapezoidal approximation to the integral.

(e) Do a von Neumann stability analysis of the FTCS method (2) on an unbounded domain. Show that if Δx, Δt 0 such that $\nu = a\Delta t/\Delta x^2$ is held fixed, the FTCS amplification factor differs from the exact amplification factor by $O(\Delta t^2)$ and show that the method is stable for $\nu < 1/2$.

(f) If $\nu > 1/2$ what is the fastest growing wavenumber, and how much does it grow in a fixed time T ($T/\Delta t$ timesteps)?

(g) Write a Matlab function that implements the FTCS scheme for this problem for arbitrary a, L, N, and ν. Solve the problem analytically and numerically for $L = 1$, $a = 1$, and $f(x) = 1 - \cos(2\pi x)$, and plot the ∞-norm of the error at time 0.016 for $N = 5, 10, 20, 40$ points with $\nu = 0.4$. Does the rate of error decrease with Δx at the rate we expect? Please submit your FTCS code as part of the solution.