

Final Assignment Solution

- 1) Consider a FEM using an arbitrary monotone increasing set of nodes $x_j, j = 0, \dots, N$, based on piecewise linear chapeau functions $\varphi_j(x)$, each of which is 1 at x_j and 0 at all other nodes:

$$q(x, t) = \sum_{j=0}^N q_j(t) \varphi_j(x).$$

The node $x_0 = 0$ is chosen to be the left boundary, so its expansion coefficient $q_0(t)$ is determined by the left BC to be $q_0(t) = \sin(50t)$. The node x_N is chosen to be the right boundary; its expansion coefficient is unknown. Let $\Delta x_{j+1/2} = x_{j+1} - x_j$. As in class, the FEM equations are found by zeroing the projection of the residual onto each basis function that has an unknown expansion coefficient:

$$0 = \langle \varphi_j(x), R(x, t) \rangle = \sum_{n=0}^N I_{jn} \frac{da_n}{dt} + \sum_{n=0}^N J_{jn} a_n, \quad j = 1, \dots, N.$$

The required inner products are easily computed for the interior nodes:

$$I_{jn} = \langle \varphi_j(x), \varphi_n(x) \rangle = \frac{1}{6} \begin{cases} \Delta x_{j-1/2} & n = j-1 \\ 2(\Delta x_{j-1/2} + \Delta x_{j+1/2}) & n = j \\ \Delta x_{j+1/2} & n = j+1 \end{cases}, \quad j = 1, \dots, N-1,$$

$$J_{jn} = \left\langle \varphi_j(x), \frac{d\varphi_n}{dx} \right\rangle = \begin{cases} -1/2 & n = j-1 \\ 0 & n = j \\ 1/2 & n = j+1 \end{cases}, \quad j = 1, \dots, N-1.$$

For the right boundary node, only the projections with the node to its left and the self-projection between $x_{N-1} < x < x_N$ contribute, altering the inner products to:

$$I_{Nn} = \langle \varphi_N(x), \varphi_n(x) \rangle = \frac{1}{6} \begin{cases} \Delta x_{N-1/2}, & n = N-1 \\ 2\Delta x_{N-1/2}, & n = N \end{cases},$$

$$J_{Nn} = \left\langle \varphi_N(x), \frac{d\varphi_n}{dx} \right\rangle = \begin{cases} -1/2, & n = N-1 \\ 1/2, & n = N \end{cases}.$$

Defining the solution vector $\mathbf{q}(t) = \{q_j(t), j = 1, \dots, N\}$, the tridiagonal inner product matrices $\mathbf{I}, \mathbf{J} = \{I_{jn}, J_{jn}, j, n = 1, \dots, N\}$, and the vectors $\mathbf{i}_0, \mathbf{j}_0 = \{I_{0n}, J_{0n}, j, n = 1, \dots, N\}$, we can write the FEM in matrix form as

$$\mathbf{I} \frac{d\mathbf{q}}{dt} + \mathbf{J}\mathbf{q} = \begin{cases} -I_{10} dq_0/dt - J_{10} q_0 & j = 1 \\ 0 & j > 1 \end{cases}.$$

Using trapezoidal time differencing, we obtain the desired time-discretized FEM:

$$\mathbf{I} \frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} + \mathbf{J} \frac{\mathbf{q}^{n+1} + \mathbf{q}^n}{2} = \begin{cases} r_{01} = -I_{10}(q_0^{n+1} - q_0^n) / \Delta t - J_{10}(q_0^{n+1} + q_0^n) / 2 & j = 1 \\ 0 & j > 1 \end{cases}$$

$$\left(\frac{\mathbf{I}}{\Delta t} + \frac{\mathbf{J}}{2} \right) \mathbf{q}^{n+1} = \left(\frac{\mathbf{I}}{\Delta t} - \frac{\mathbf{J}}{2} \right) \mathbf{q}^n + \begin{cases} r_{01} & j = 1 \\ 0 & j > 1 \end{cases}$$

The Matlab script finalp1.m on the class web page implements this tridiagonal system for time-marching the advection equation with the specified IC and BC. The first part uses the matrices \mathbf{I} and \mathbf{J} computed for constant grid spacing $\Delta x = 0.01$ (problem 1); the second part recalculates these matrices for the stretched grid $x_j = 1 - (1 - \xi_j)^{1/2}$, $\xi_j = 0.02j, j = 1, \dots, 50$.

Fig. 1 compares these two solutions at $t = 0.9$ with the exact solution $q(x, t) = \sin[50 \max(t - x, 0)]$. Both do a decent job near the left boundary, where the wave is being forced and where even the stretched grid has a resolution sufficient to resolve the wave well. The uniform-grid solution is quite respectable throughout the domain, but the stretched-grid solution starts to degrade near the right boundary where it no longer has sufficient resolution to adequately resolve a wave of wavelength $2\pi/50$.

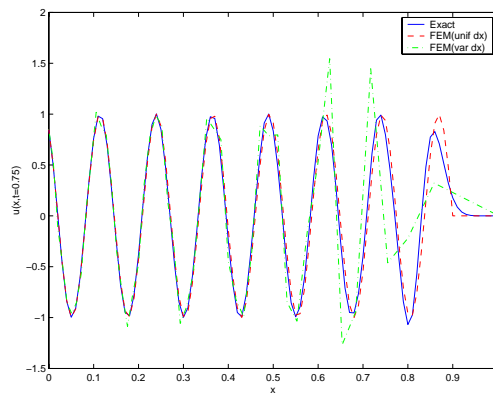


Fig. 1. FEM solutions to advection equation with uniform and stretched grids.

2. Fig. 2, generated following the instructions in the class handout, shows the $t = 2$ FEM solution to the wave equation IBVP generated by the Matlab PDE toolbox:

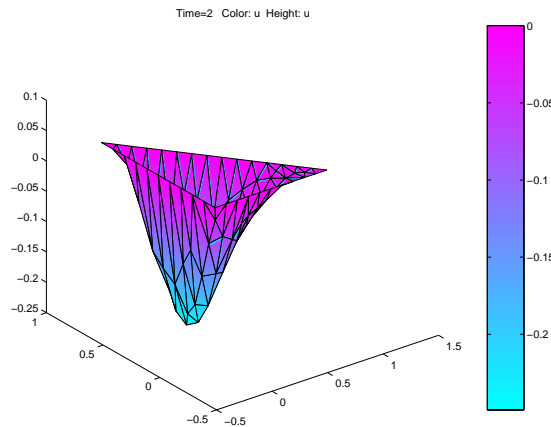


Fig. 2