Final Project (Please do by yourself without consulting with anyone)

1. Consider the 1D advection equation:

$$q_t + q_x = 0, \quad 0 < x < 1, \quad t > 0,$$

with initial condition $q(x, 0) = 0$ and boundary condition $q(0, t) = \sin(50t)$.

(a) Discretize this problem with a finite element method like in class, using piecewise linear chapeau basis functions and trapezoidal time differencing, a grid spacing $\Delta x = 0.01$, and a timestep $\Delta t = 0.005$. Note that the left boundary condition specifies the coefficient of the basis function centered on the left boundary, so the unknowns are the coefficients $a_j(t)$ of the basis functions centered at the gridpoints $x_j = j\Delta x$, $j = 1,\ldots,100$, corresponding to the interior gridpoints and the right boundary. Using a sparse solver, solve the resulting tridiagonal system and overplot the numerical solution $q(x, 0.9)$ on the exact solution at this time. How well do they compare?

(b) Now discretize the problem using exactly the same kind of finite element method, but on a nonuniform grid $x_j = 1 - (1 - \xi_j)^{1/2}$, $\xi_j = 0.02j$, $j = 1,\ldots,50$. This grid starts with the same grid spacing as in part (a) near x = 0, but the grid stretches greatly as we move close to the right boundary $x = 1$. Using trapezoidal time differencing and the same timestep as before, again calculate the numerical solution $q(x, 0.9)$ and overplot it on the result of part (a). Does the stretched grid affect the numerical solution? Why or why not?

2. Use the Matlab PDE Toolbox to solve the following initial-boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

in the right-triangular domain Ω: $\{x, y > 0, 0 < x + y < 1\}$ over the time interval $0 < t < 2$ with IC $u(x, y, 0) = 0$ and BCs $u(x+y=1, t) = u(x, 0, t) = 0$, $u(0, y, t) = \sin(\pi y)\sin(\pi t)$. Use the default grid generated by the toolbox. To show me you got this to work, please print out a 3D surface plot of the solution at $t = 2$ with the grid shown.