Chapter 1

Chapter 1

1.5 Prove that exactly half the area of the earth lies equatorward of 30° latitude.
Solution:

$$\begin{aligned}
Area &= \int_0^{2\pi}\int_{-30^\circ}^{30^\circ} R_E^2 \cos\phi d\phi d\lambda \\
&= 2\pi R_E^2 \int_{-30^\circ}^{30^\circ} \cos\phi d\phi \\
&= 2\pi R_E^2 \sin\phi|_{-30^\circ}^{30^\circ} \\
&= 2\pi R_E^2 (0.5 - (-0.5)) \\
&= 2\pi R_E^2, \text{ half the area of a sphere.}
\end{aligned}$$

1.6 How many days would it take a hot air balloon traveling eastward along 40°N at a mean speed of 15 m s^{-1} to circumnavigate the globe?

Solution: The length of the latitude circle is $l = 2\pi R_E \cos 40^\circ = 2\pi \times 6.37 \times 10^6 \times 0.766 = 30.7 \times 10^6$ m or 30,700 km.

The time is the distance divided by the velocity, i.e.,

$$t = \frac{l}{v} = \frac{30.7 \times 10^6 \text{ m}}{15 \text{ m s}^{-1}} = 2.05 \times 10^6 \text{ s} = 23.7 \text{ days}$$

1.7 Prove that pressure expressed in cgs units of millibars (mb = 10^{-3}b) is numerically equal to pressure expressed in SI units of hPa (hectoPascals $=10^2$ Pa).

Solution:

$$\begin{aligned}
1 \text{ mb} &= 10^3 \frac{\text{dynes}}{\text{cm}^{-2}} = 10^3 \frac{\text{g cm}}{\text{s}^{-2} \text{ cm}^{-2}} = 10^3 \frac{\text{g}}{\text{s}^{-2} \text{ cm}} \\
&= 10^3 \frac{10^{-3} \text{ kg}}{\text{s}^{-2}\ 10^{-2} \text{ m}} = \frac{10^2 \text{ kg m}}{\text{s}^{-2} \text{ m}^2} = 10^2 \frac{\text{N}}{\text{m}^{-2}} = 1 \text{ hPa}
\end{aligned}$$

1.8 How far below the surface of the water does a diver experience a pressure of 2 atmospheres (i.e., a doubling of the ambient atmospheric pressure due to the weight of the overlying water). Answer ∼10 m.

Solution: At the depth where the pressure is 2 atm, half the pressure is due to the weight of the overlying air and the other half (1 atm) is due to the weight of the overlying water, which is equal to the density times the depth times g . The density of water is ρ =1000 kg/m^3 .

$$\begin{aligned} p &= \rho g z = 10^5\text{Pa}; \\ z &= 10^5/(10^3 \times 9.8) \simeq 10 \text{ m} \end{aligned}$$

1.9 In a sounding taken on a typical winter day at the South Pole the temperature at the ground is -80°C and the temperature at the top of a 30 m high tower is −50°C. Estimate the lapse rate within the lowest 30 m, expressed in K km^{-1}.

Solution:

$$\Gamma \equiv -\frac{\triangle T}{\triangle z} = -\frac{(-50-(-80))}{30} = -1°\text{C m}^{-1} = -1,000°\text{C km}^{-1}, \text{ a very strong inversion!!}$$

1.10 "Cabin altitude" in typical commercial airliners is around 1.7 km. Estimate the typical pressure and density of the air in the passenger cabin.

Solution: Let us assume values at sea-level of $p = 1,000$ hPa and $\rho = 1.25$ kg m^{-3} and exponential relationships of the form

$$p = p_0 e^{-z/H}$$

and

$$\rho = \rho_0 e^{-z/H}$$

where the scale height $H = 7.5$ km. Substituting $z = 1.7$ km, we obtain $p \sim 800$ hPa and $\rho \sim 1.00$ kg m^{-3}.

1.11 Prove that density and pressure, which decrease more or less exponentially with height, decrease by a factor of 10 over a depth of ln 10 = 2.303 times the scale height.

Solution: If $p = p_0 e^{-z/H}$,then $p/p_0 = 0.1$ at the level where $e^{-z/H} = 0.1$ or $z = -H \ln(0.1)$ or $z = H \ln(10) = 2.303H$.

1.12 Consider a perfectly elastic ball of mass m bouncing up and down on a horizontal surface under the action on a downward gravitational acceleration g. Prove that in the time average over an integral number of bounces, the downward force exerted by the ball upon the surface is equal to the weight of the ball. [**Hint:** The downward force is equal to the the downward momentum imparted to the surface with each bounce divided by the time interval between successive bounces.] Does this result suggest anything about the "weight" of an atmosphere comprised of gas molecules?

Solution: The momentum imparted to the surface with each bounce is $\triangle p = 2mv_0$.The time required for the ball to fall from the top of its orbit to the surface is $t_1 = v_0/g$, so the time between bounces must be $\triangle t = 2v_0/g$. The average downward force is equal to the average rate of transfer of momentum $F = \frac{\triangle p}{\triangle t} = 2mv_0/\left(2v_0/g\right) = mg$, which is the weight of the ball.

1.13 Estimate the percentage of the mass of the atmosphere that resides in the stratosphere based on the following information. The mean pressure level of tropical tropopause is around 100 hPa and that of the extratropical tropopause is near 300 hPa, where the break between the tropical and extratropical tropopause occurs near 30° latitude, in which case, exactly half the area of the earth lies in the tropics and half in the extratropics. On the basis of an inspection of Fig. 1.7, verify that these estimates are reasonably close to observed conditions.

Solution: The boundary between the tropical tropopause and the extratropical tropopause is close to 30° latitude. Roughly half the mass of the atmosphere lies equatorward of this latitude. In the tropical half of the globe the tropopause level is close to 100-hPa. At any height, pressure is the weight of the atmosphere above it (per unit area), so we can estimate the ratio r of stratospheric to total mass as $r = \frac{m_{strat}}{m_{tot}} = \frac{p_{strat}}{p_0}$, where p_0 is the surface pressure (the mass above the stratopause is negligible). In the tropics $r = \frac{100}{1000} \sim 0.1$. Roughly 10% of the mass of the atmosphere lies in the stratosphere . In the extratropical half of the globe the tropopause lies close to 300-hPa, so roughly 30% of the mass of the atmosphere lies in the stratosphere. So for the globe as a whole the fraction of the mass of the astmosphere that lies in the stratosphere is the mean of these two values, or ~20%.

1.14 If the earth's atmosphere consisted of an incompressible fluid whose density was everywhere equal to that observed at sea level (1.25 kg m^{-3}) how deep would it have to be to account for the observed mean surface pressure of ~10^5 Pa.

Solution: If h is the height of the free surface of the fluid

$$p = \rho g h = 1000 \text{ hPa}$$

Solving, we obtain

$$h = \frac{10^5}{9.8 \times 1.25} \simeq 8,000 \text{ m}$$

1.15 The mass of the water vapor in the atmosphere is equivalent to that of a layer of liquid water how deep? *Answer:* ~ 2.4 cm.

Solution: Averaged over the globe, water vapor accounts for 0.24% of the mass of the atmosphere. Gence its mass per unit area is 0.24% of 1.017 $\times 10^4$ kg m^{-2} or 24 kg m^{-2}. The density of liquid water is 10^3 kg m^{-3}. Hence a layer of water 0.024 m or 2.4 cm is equivalent to the mass of water vapor in the atmosphere.

1.16 If the density of air decreases exponentially with height from a value of 1.25 kg m^{-3} at sea-level, calculate the scale height that is consistent with the observed global mean surface pressure of ˜10^5 hPa. [**Hint**: Integrate the counterpart of (1.4) from the earth's surface to infinity to obtain the atmospheric mass per unit area.]

Solution: Integrating (1.4) from sea-level to the top of the atmosphere assuming a constant value of the scale height yields

$$\begin{aligned} p_0 &= \int_0^\infty \rho g dz \\ &= \int_0^\infty \rho_0 e^{-z/H} g dz \\ &= \rho_0 g \int_0^\infty e^{-z/H} dz \\ &= \rho_0 g H e^{-z/H}|_0^\infty \\ &= \rho_0 g H \end{aligned}$$

Solving, we obtain

$$\begin{aligned} H &= \frac{p_0}{\rho_0 g} \\ &= \frac{10^5}{(1.25 \times 9.8)} \\ &\sim 8,000 \text{ m} \end{aligned}$$

1.17 The equatorward flow in the tradewinds is on the order of 1 m s^{-1} averaged around the circumference of the earth at 15°N and 15°S, and it extends through a layer extending from sea-level up to around the 850 hPa pressure surface. Estimate the equatorward mass flux into the equatorial zone due to this circulation. **Hint**: The equatorward mass flux across the 15°N, in units of kg s^{-1} is given by

$$-\oint_{15^\circ\mathrm{N}} \int_0^{z_{850}} \rho v dx dz$$

where is the density of the air, v is the meridional (northward) velocity component, the line integral denotes an integration around the 15°N latitude circle and the vertical integral is from sea-level up to the height of the 850 hPa surface.]

Solution: We evaluate the integral, make use of the relations

$$\oint_{15^\circ\mathrm{N}} dx = 2\pi R_E \cos 15^\circ (= l, \text{ the length of the latitude circle})$$

and

$$\int_0^{z_{850}} \rho dz = -dp/g = \frac{p_0 - p_{850}}{g} \ (= m, \text{ the mass per unit area in the layer })$$

The mass flux, per unit length along the latitude circle is mv. Multiplying by the length of the latitude circle and by 2 (the flux enters the equatorial region from 15°S and 15°N) we obtain the total mass flux

$$\begin{aligned} F_m &= 2 \times l \times m \times v = \\ &= 2 \times 2\pi \times 6.37 \times 10^6 \times \cos(15^\circ) \times 1.5 \times \frac{10^4}{9.8} \times 1 \\ &= 11.8 \times 10^{10} \text{ kg s}^{-1}. \end{aligned}$$

1.18 During September, October and November the mean surface pressure over the northern hemisphere increases at a rate of ∼1 hPa per 30-day month. Calculate the mass averaged northward velocity across the equator that is required to account for this pressure rise. [**Hint**: Assume that atmospheric mass is conserved: i,e, that the pressure rise in the northern hemisphere is entirely due to the influx of air from the southern hemisphere. By defining the mass averaged northward velocity

$$v_m \equiv \frac{\oint \int_0^\infty \rho v dx dz}{\oint \int_0^\infty \rho dx dz}$$

we can write the mass flux in the form

$$F_m = v_m \oint \int_0^\infty \rho dx dz$$

which can be solved for v_m.]

Solution: The required mass transport F_m is equal to the rate of pressure change, divided by g, times the area of the hemisphere ($2\pi R_E^2$), , or

$$F_m = \frac{1}{g} \times \frac{\delta p}{\delta t} \times 2\pi R_E^2 = v_m \oint \int_0^\infty \rho dx dz$$

where δt is 1 month or 2.59×10^6 s. Noting that $\oint dx = 2\pi R_E$, and $\int_0^\infty \rho dz = p_0/g$, we can write

$$\frac{\delta p}{g \times \delta t} \times 2\pi R_E^2 = 2\pi R_E \times v_m \times \frac{p_0}{g}$$

Solving, we obtain .

$$\begin{aligned} v_m &= \frac{\delta p \times R_E}{p_0 \delta t} \\ &= 100 \times \frac{6.37 \times 10^6}{10^3 \times 2.59 \times 10^6} \\ &= 2.46 \text{ mm s}^{-1} \end{aligned}$$

Chapter 2

Chapter 2

2.9 In the oceanographic literature, the unit of mass transport is the sverdrup (Sv) (millions of cubic meters per second).

The transport of ocean water by the Gulf Stream is estimated to be on the order of 150 Sv. Compare the transport by the Gulf Stream with the transport by the tradewinds estimated in the Exercise 1.16.

1 Sv=10^6m^{3}s^{-1}. The density of water is ρ =10^3kg m^{-3}.Therefore the mass transport Tr by the Gulf Stream is

$$Tr = 150 \times 10^6 \times 10^3 \text{kg s}^{-1} = 1.5. \times 10^{11} \text{ kg s}^{-1}.$$

Comparing this result with the transport by the trade winds (In the book's answer of Exercise 1.16, the estimate was about 1.3 $\times 10^{11}$ kg s^{-1}) , we see that the two transports are of same order of magnitude.

2.10 If the air flowing equatorward in the tradewinds in Exercise 1.16 contains 20 g of water vapor per kg of air, estimate the mean rainfall rate within the equatorial zone (15°N - 15°S) attributable to this transport.

[**Hint**: For purposes of this problem the latitude belt between 15°N and 15°S can be treated as a cylinder.]

The volume of water entering the region is given by the air transport ($T_r = 1.32 \times 10^{11}$kg s^{-1}) times the mass of water per unit mass of air ($q = 20$ g kg^{-1})

divided by the density of water ($\rho = 10^3$ kg m^{-3}). Assuming that all the water entering becomes rain, the mean rainfall rate R_r over the area A is:

$$R_r = \frac{q \times Tr}{\rho \times A}$$

If we treat the surface between 15°N and 15°S as a cylinder, the area A is:

$$\begin{aligned} A &= 2 \times \pi \times R_E \times d, \quad \text{where } d = 2 \times R_E \times \sin(15°); \\ A &= 4 \times \pi \times R_E^2 \sin(15°). \end{aligned}$$

Therefore

$$\begin{aligned} R_r &= \frac{20 \times 10^{-3} \times 1.32 \times 10^{11}}{10^3 \times 4 \times \pi \times (6.37)^2 \times 10^{12} \times \sin(15)} \\ &= \frac{20 \times 1.32}{4 \times \pi \times (6.37)^2 \times \sin(15)} \times 10^{-7} \\ &= 0.2 \times 10^{-7} \text{m s}^{-1} = 1.7 \text{ mm day}^{-1}. \end{aligned}$$

2.11 Reconcile the mass of oxygen in the atmosphere in Table 2.4 with the volume concentration given in Table 1.1.

In Table 2.4 we read that the mass of oxygen per unit area is $m_{O_2} = 2.353 \times 10^3$ kg m^{-2}

Using the relation

$$m_{O_2} = \frac{c_{O_2} \times M_{O_2}}{\sum c_i M_i} \times m_a$$

with m_a = mass of the atmosphere (per unit area) and M_i = molecular weight of the element i,

we can find the corresponding volume concentration c_{O_2} :

$$\begin{aligned} c_{O_2} &= \frac{m_{O_2}}{m_a} \frac{\sum c_i M_i}{M_{O_2}} \\ &= \frac{2.353 \times 10^3}{1.017 \times 10^4} \times \frac{28.97}{32} \\ &= 0.2095 \end{aligned}$$

which is the value given in Table 1.1.

2.12 The current rate of consumption of fossil fuels is 7 Gt C per year. Based on the data in Table 2.3, how long would it take to deplete the entire fossil fuel reservoir of fossil fuels (a) if consumption continues at the present rate and (b) if consuption increases at a rate of 1% per year over the next century and remains constant thereafter.

The quantity of carbon in fossil fuels given in Table 2.3 is m =10 kg m^{-2} (averaged over the Earth's surface).

The total mass of carbon in fossil fuels over the Earth is $m_{tot} = m \times 4\pi R_E^2$ =5099 Gt

a) if consumption continues at the present rate r_o(=7 Gt C per year), the time required for depletion t_d is:

$$\begin{aligned} t_d &= \frac{m_{tot}}{r_o} = \frac{10 \times 4\pi \times (6.37)^2 \times 10^{12}}{7 \times 10^{12}} \\ &= \frac{4\pi \times (6.37)^2}{7} \times 10 = 728 \text{ years.} \end{aligned}$$

b) if consumption increases at a rate of 1% per year over the next century:

$$r = r_o e^{0.01t}$$

Substituting t =100, we find the rate of consumption at t=100 year: $r_{100} = r_o e$ =19 Gt per year.

The total consumption until year 100 is:

$$\begin{aligned} C_{100} &= \int_0^{100} r_o e^{0.01t} dt \\ &= \frac{r_o}{0.01} \times \left[e^{0.01t}\right]_0^{100} \\ & \quad r_o \times 100 \times (e-1) \\ &= 7 \times 100 \times (e-1) \\ &= 1203 \text{ Gt.} \end{aligned}$$

The carbon left after 100 years is $m_{tot} - C_{100}$; it is depleted at the constant rate r_{100} in a time

$$\begin{aligned} t &= \frac{m_{tot} - C_{100}}{r_{100}} \\ &= 205 \text{ years} \end{aligned}$$

The total time of depletion is t_d = 100+205= 305 years.

2.13 If all the carbon in the fossil fuel reservoir in Table 2.3 were consumed, and if half of it remained in the atmosphere in the form of CO_2, by what proportion would the atmospheric concentration of CO_2 increase relative to current values? By what proportions would the atmospheric O_2 concentration decrease?

Half of the quantity of carbon in fossil fuels is 5 kg m^{-2},which is added to the present quantity of carbon in atmospheric CO_2(1.6 kg m^{-2}).

Using the relation

$$m_C = \frac{c_{CO_2} \times M_C}{\sum c_i M_i} \times m_a$$

with m_a = mass of the atmosphere (per unit area) and M_i = molecular weight of the element i,

we can find the corresponding volume concentration c_{O_2} :

$$\begin{aligned} c_{CO_2} &= \frac{m_C}{m_a} \frac{\sum c_i M_i}{M_C} \\ &= \frac{(5+1.6)}{1.017 \times 10^4} \times \frac{28.94}{12} \\ &= 1.57 \times 10^{-3} = 1570 \text{ ppmv} \end{aligned}$$

The increase in CO_2 implies a decrease in O_2 (for each molecule of C, one molecule of O_2 is needed to form one molecule of CO_2).

Therefore (1570 - 380) ppmv of oxygen would disappear. The decrease of O_2 concentration would be of order: $\frac{(1570-380)\times 10^{-6}}{0.2095} \sim 0.6\%$.

2.14 Before the US/USSR hydrogen bomb tests of the 1950's, one out of every x molecules of atmospheric CO_2 was ^{14}C, whose half life is 5,730 years. Estimate the abundance of ^{14}C remaining in a 50,000 year old sample.

A decay in time is represented by an exponential relation:

$$c = c_o e^{-\frac{t}{\tau}}$$

where c_o is the initial concentration, τ is the $e - folding$ time (time to decrease by a factor e). To find the relation between e-folding time

and *half-life* (t_{hl}), we impose:

$$\begin{aligned} \frac{c}{c_o} &= \frac{1}{2} = e^{-\frac{t_{hl}}{\tau}} \\ t_{hl} &= \ln(2) \times \tau \end{aligned}$$

Therefore in our example $\tau = t_{hl}/\ln(2) = 8267$ years. The concentration after 50000 years is

$$\begin{aligned} c &= c_o e^{-\frac{t}{\tau}} \\ &= c_o e^{-\frac{50000}{8267}}; \\ \frac{c}{c_o} &= 2.36 \times 10^{-3} \end{aligned}$$

Alternatively, we could solve this problem using the equivalent relation:

$$c = c_o 2^{-\frac{t}{t_{hl}}}$$

2.15 If all the carbon in the inorganic and organic sedimentary rock reservoirs in Table 2.3 were in the atmosphere instead, in the form of CO_2, together with the atmosphere's present constituents, what would the mean surface pressure be? What would be the volume concentration of N_2?

The quantity of carbon in sedimentary rocks is: m_c =100,000 kg m^{-2}. If this carbon were in the atmosphere instead, the atmospheric

mass would be dominated by carbon (the present mass is only: m_a =1.017 $\times 10^4$ kg m^{-2}). We need to include the mass of oxygen to

find the mass of CO_2 m_{CO_2} (for one molecule of C, one molecule of $CO_2$is formed)

$$m_{co_2} = m_c \times \frac{M_{CO_2}}{M_c}$$

The new atmospheric mass would be $m_{new} = m_a + m_{co_2}$ =1.017 $\times 10^4$ +100,000 $\times$44/12= 3.77$\times 10^5$kg m^{-2}($\sim$37 times m_a). The new

atmospheric pressure would be $p_{new} = m_{new} g \sim 37\times 10^5$Pa.

The mass of N_2 is constant:

$$\begin{aligned} m_{_{N_2}} &= \frac{c_{_{N_2}} \times M_{_{N_2}}}{\sum c_i M_i} \times m_a \\ \frac{c_{_{N_2}}}{\sum c_i M_i} \times m_a &= \frac{(c_{_{N_2}})_{new} \mu}{(\sum c_i M_i)_{new}} \times m_{new} \end{aligned}$$

Therefore the new fractional concentration of N_2 would be :

$$\begin{aligned} (c_{_{N_2}})_{new} &= c_{N_2} \times \frac{m_a}{m_{new}} \times \frac{(\sum c_i M_i)_{new}}{\sum c_i M_i} \\ &\sim 0.78 \times \frac{1}{37} \times \frac{44}{29} \sim 3\% \end{aligned}$$

(assuming that the new apparent molecular weight of the atmosphere $\sim$ M$_{_{CO_2}}$)

Chapter 3

Chapter 4

4.12 Remote sensing in the microwave part of the spectrum relies on radiation emitted by oxygen molecules at frequencies near 55 gHz. Calculate the wavelength and wavenumber of this radiation.

From (4.2)

$$\lambda = \frac{c^*}{\widetilde{\nu}} = \frac{3 \times 10^8 \text{ m s}^{-1}}{55 \times 10^9 \text{ s}^{-1}} = 5450 \ \mu\text{m}$$

4.13 The spectrum of monochromatic intensity can be defined either in terms of wavelength λ or wavenumber ν such that the area under the spectrum, plotted as a linear function of λ or ν is proportional to intensity. Show that $I_\nu = \lambda^2 I_\lambda$.

$$dI = I_\lambda d\lambda = I_\nu d\nu$$

From (4.1)

$$\nu = \frac{1}{\lambda}$$

from which it follows that

$$d\nu = -\frac{d\lambda}{\lambda^2}$$

Substituting for $d\lambda$ in the first expression, cancelling the common factor $d\nu$, and ignoring the minus sign, which is taken into account by

reversing the direction of the integration, we obtain

$$I_\nu = \lambda^2 I_\lambda$$

4.14 A body is emitting radiation with the following idealized spectrum of monochromatic flux density

$$
\begin{array}{cc}
\lambda < 0.35\ \mu\text{m} & F_\lambda = 0 \\
0.35\ \mu\text{m} < \lambda < 0.50\ \mu\text{m} & F_\lambda = 1.0\ \text{W m}^{-2}\mu\text{m}^{-1} \\
0.50\ \mu\text{m} < \lambda < 0.70\ \mu\text{m} & F_\lambda = 0.5\ \text{W m}^{-2}\mu\text{m}^{-1} \\
0.70\ \mu\text{m} < \lambda < 1.00\ \mu\text{m} & F_\lambda = 0.2\ \text{W m}^{-2}\mu\text{m}^{-1} \\
\lambda > 1.00\ \mu\text{m} & F_\lambda = 0
\end{array}
$$

Calculate the flux density of the radiation.

$$
\begin{aligned}
F &= \int F_\lambda d\lambda = \sum_{i-1}^{N} F_{\lambda i} \triangle\lambda_i \\
&= 1.0 \times 0.15 + 0.5 \times 0.20 + 0.2 \times 0.3 \\
&= 0.15 + 0.10 + 0.06 = 0.31\ \text{W m}^{-2}
\end{aligned}
$$

4.15 An opaque surface with the following absorption spectrum is subjected to the radiation described in the previous exercise.

$\lambda < 0.70\ \mu$m $\quad A_\lambda = 0$

$\lambda > 0.70\ \mu$m $\quad A_\lambda = 1$

How much of the radiation is absorbed? How much is reflected?

$$
\begin{aligned}
F(\text{absorbed}) &= \int A_\lambda F_\lambda d\lambda = \sum_{i-1}^{N} A_{\lambda i} F_{\lambda i} \triangle\lambda_i \\
&= 1.0 \times 1.0 \times 0.15 + 1.0 \times 0.5 \times 0.2 \\
&= 0.15 + 0.10 = 0.25\ \text{W m}^{-2}
\end{aligned}
$$

$$
\begin{aligned}
F(\text{reflected}) &= \int (1 - A_\lambda) F_\lambda d\lambda = \sum_{i-1}^{N} (1 - A_{\lambda i}) F_{\lambda i} \triangle\lambda_i \\
&= 1.0 \times 0.3 \times 0.2 \\
&= 0.06\ \text{W m}^{-2}
\end{aligned}
$$

4.16 Calculate the ratios of the incident solar radiation at noon on north and south facing 5° slopes (relative to the horizon) in seasons in which the solar zenith angle is (a) 30° and (b) 60°.

For the 30° solar zenith angle the ratio is

$$
r = \frac{I \cos 35^\circ}{I \cos 25^\circ} = 0.84
$$

and for the 60° zenith angle the ratio is

$$
r = \frac{I \cos 65^\circ}{I \cos 55^\circ} = 0.74
$$

4.17 Compute the daily insolation at the North Pole at the time of the summer solstice when the earth-sun distance is 1.52 $\times 10^8$ km. The tilt of the earth's axis is 23.5°. Compare this value with the minimum value that occurs in association with the earth's orbital cycles described in §2.5.3. *Answer*: 46.4 MJ m^{-2} day^{-1} versus xxx MJ m^{-2} day^{-1}.

4.18 Compute the daily insolation at the top of the atmosphere at the equator at the time of the equinox (a) bu integeating the flux density over a 24 hour period and (b) by simple geometric considerations. Compare your result with the value in the previous exercise and with Fig. 4.xx. *Answer:* 38.0 MJ m^{-2} day.

4.19 What fraction of the flux of energy emitted by the sun does the earth intercept? *Answer* 1.4 $\times 10^{-10}$.

4.20 Show that for small variations in the earth's radiation balance

$$\frac{\delta T_E}{T_E} = \frac{1}{4}\frac{\delta F_E}{F_E}$$

where T_E is the planet's equivalent blackbody temperature and F_E is the flux of radiation emitted from the top of its atmosphere. Use this relationship to estimate the change in effective temperature that would occur in response to (a) the seasonal variations in the sun-earth distance due to the eccentricity of the earth's orbit (presently $\sim$ 3%),(b) an increase in the earth's albedo from 0.30 to 0.31.

From (4.12)

$$F = \sigma T^4$$

Taking the log yields

$$\ln F = 4 \ln T$$

Taking the differential yields

$$\frac{\delta F}{F} = 4\frac{\delta T}{T}$$

and dividing both sides by 4 yields

$$\frac{\delta T_E}{T_E} = \frac{1}{4}\frac{\delta F_E}{F_E} \tag{1}$$

(a) From the inverse square law

$$F_E = const \times d^{-2}$$

where d is the earth-sun distance. Taking the log yields

$$\ln F_E = \ln const - 2\ln d$$

Taking the differential yields

$$\frac{\delta F_E}{F_E} = -2\frac{\delta d}{d} \tag{2}$$

Combining (1) and (2) yields

$$\frac{\delta T_E}{T_E} = -\frac{1}{2}\frac{\delta d}{d} \tag{1}$$

Substituting $\delta d/d = 0.03$ yields $\delta T_E/T_E = -0.06$. If $T_E = 255$ K, then $\delta T_E = 1.5$ K.

(b)

$$F_E = const \times (1 - A)$$

where A is the eplanetary albedo. Taking the log yields

$$\ln F_E = \ln const - \ln(1 - A)$$

Taking the differential yields

$$\frac{\delta F_E}{F_E} = \frac{\delta(1 - A)}{(1 - A)} \tag{2}$$

Combining (1) and (2) yields

$$\frac{\delta T_E}{T_E} = \frac{1}{4}\frac{\delta(1 - A)}{(1 - A)} \tag{3}$$

Substituting $\delta d/d = 0.01/0.70$ yields $\delta T_E/T_E = -0.00357$. If $T_E = 255$ K, then $\delta T_E = 0.91$ K.

4.21 Show that the emission the flux density of incident solar radiation on any planet in our solar system is 1366 $\mathrm{Wm}^{-2}\times\ d^{-2}$, where d is the Planet-Sun distance, expressed in astronomical units (A.U., multiples of the Earth-Sun distance).

Method 1: The flux density is equal to the intensity of solar radiation I_s (the same for all planets) times the arc of solid angle $\delta\Omega$ subtended by the sun, as viewed from the planet, i.e.,

$$F = I_s \delta\Omega$$

where

$$\delta\Omega = 4\pi \times \left(\frac{R_s}{d}\right)^2$$

where R_s is the radius of the sun and d is the distance between the planet and the sun. Hence,

$$F = (4\pi I_s R_s^2) d^{-2}$$

where the factor in parentheses is the same for all planets. For Earth,

$$1366 = (....)d_E^{-2} \tag{1}$$

and for any other planet

$$F_p = (....)d_p^{-2} \tag{2}$$

Dividing (2) by (1) yields

$$\frac{F_p}{1366} = \left(\frac{d_p}{d_E}\right)^{-2} \tag{3}$$

Method 2: The final result above is an expression of the inverse square law which follows directly from the fact that the flux of solar

radiation through all spheres concentric with the sun must be the same and hence the product of the flux density of solar radiation

times distance from the sun must be the same for all planets.

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4.22 Estimate the flux density of the radiation emitted from the top of the sun's atmosphere using two different approaches: (a) starting with the intensity and making use of the results in Exercise 4.3 and (b) making use of the relationship derived in the previous problem. (c) estimate the output of the sun in watts.

(a) The intensity of solar radiation as given in Exercise 4.2 is $I_S = 2.00 \times 10^7$ W m^{-2} sr^{-1}. In Exercise 4.3 it is shown that $F = \pi I$. Hence,

$$F = \pi \times 2.00 \times 10^7 = 6.28 \times 10^7 \text{ W m}^{-2}.$$

(b)

$$F = 1366\left(\frac{d}{R_s}\right)^2 = 1366\left(\frac{1.50 \times 10^{11}}{7.00 \times 10^8}\right)^2 = 6.28 \times 10^7 \text{ W m}^{-2}$$

(c) The flux of energy emitted by the sun is

$$4\pi R_s^2 F = 4\pi \times (7.00 \times 10^8)^2 \times 6.28 \times 10^7 = 3.87 \times 10^{26} \text{ W}$$

4.23 By differentiating (4.10) derive Wien's displacement law. (Hint: throughout most of the wavelength range of the blackbody spectrum, the exponential term in the denominator is much larger than 1).

4.24 Show that for very long wavelengths, the Planck monochromatic intensity $B(T)$ is linearly proportional to absolute temperature.

4.25 Use the relationship derived in Problem 4.19 to check the numerical values of T_E in Table 4.1.

4.26 The observed equivalent blackbody temperature of Jupiter is 125 K, 20 K higher than the value in Table 4.1. Assuming that the temperature of

Jupiter is in a steady state, estimate the flux density of radiation emitted from the top of its atmosphere that is generated internally by processes on the planet. *Answer*: 7 W m^{-2}.

The flux of radiation from Jupiter is

$$F = \sigma\,(125)^4 = 13.84 \text{ W m}^{-2}$$

The emission required to balance the incoming solar radiation is

$$F = \sigma\,(105)^4 = 6.89 \text{ W m}^{-2}$$

The difference 6.95 W m^{-2} must be generated by processes on the planet.

4.27 If the absorptivity of an optically thin layer of the atmosphere is A_λ for radiation passing through it at zero zenith angle and the radiation is isotropic, calculate the fraction of the flux density that is absorbed in its passage through the layer. *Answer:* $2A_\lambda$

If the layer is thin, then the fraction of the radiation that is absorbed in passing through the layer must be directlt proportional to the zenith angle ϕ.Hence, the fraction of the flux density that is absorbed is given by

$$\frac{\int_{2\pi} I A_\lambda \sec\phi \cos\phi d\Omega}{\int_{2\pi} I \cos\phi d\Omega} = \frac{2\pi I A_\lambda}{\pi I} = 2A_\lambda$$

4.28 Estimate the flux density of moonlight on a horizontal surface on Earth, assuming that the Moon has an albedo of 0.10, that it subtends the same arc of solid angle in the sky that the sun does, and that it is directly overhead. Assume that the the moon scatters sunlight isotropically. It follows that when the moon is full, the intensity of scattered light is uniform over the lunar disk. (Proving this could be posed as a separate problem.)

4.29 The *radiative relaxation time* of a planetary atmosphere is the time that would be required for its absolute temperature to decrease by a factor of e owing to the emission of blackbody radiation to space, in the absence of heat sources and sinks. Show that the radiative relaxation time is given by

$$\frac{c_p p_0}{g}$$

4.40 For incident parallel beam solar radiation in an isothermal atmosphere in which k_λ is independent of height, (a) show that optical depth is linearly proportional to pressure and (b) the heating rate (per unit mass) propotional to the strength if the incident radiation and is therefore strongest at the top of the atmosphere where the incident radiation is undepleted. Assume that the zenith angle is zero.

(a) From (4.16) for overhead radiation

$$\ln F_{\lambda\infty} - \ln F_\lambda = \int_z^\infty k_\lambda \rho r dz = \tau_\lambda$$

and if k_λ and r are independent of height we can write

$$\ln I_{\lambda\infty} - ln I_\lambda = k_\lambda r \int_z^\infty \rho dz = \tau_\lambda$$

From the hydrostatic equation

$$\int_z^\infty \rho dz = \frac{p}{g}$$

Hence,

$$\tau_\lambda = \frac{k_\lambda r p}{g} \approx \; const \times \; p$$

(b) From (4.38) the heating rate is given by

$$\left(\frac{dT}{dt}\right)_\nu = -\frac{1}{\rho c_p}\frac{dF_\lambda(z)}{dz}$$

Rewriting (4.17) to parallel beam radiation yields

$$dF_\lambda = -F_\lambda \rho r k_\lambda dz$$

Substituting into the expression for heating rate, we obtain

$$\left(\frac{dT}{dt}\right)_\nu = \frac{1}{\rho c_p} F_\lambda \rho r k_\lambda = \frac{r k_\lambda}{c_p} F_\lambda = const \times F_\lambda$$

Since F_λ decreases monotonically with depth, it follows that the heating rate must be highest at the top of the atmosphere
where the incident radiation is undepleted.

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Chapter 4

Chapter 10

10.7 Suppose that the global atmosphere and a 50 m deep ocean mixed layer both warm at a rate of 3 K per century. In the global energy balance in Fig. 10.2, by how much would the net incoming solar radiation have to exceed the outgoing earth radiation at the top of the atmosphere?

Solution: The net downward flux at the top of the atmosphere per unit are must equal the rate at which energy stored in the atmosphere and ocean mixed layer, i.e.,

$$F = \left(\frac{c_p p_0}{g} + A c_w \rho_w H \right) \frac{dT}{dt}$$

where c_p and c_w are the specific heats of air and water, respectively, g is the gravitational acceleration, p_0 is sea-level pressure, A is the fractional area of the earth's surface covered by oceans, H is the mixed layer depth, T is temperature and t is time. Substituting numerical values

$$\begin{aligned} F &= \left(\frac{1004 \times 10^5}{9.8} + 0.7 \times 4187 \times 10^3 \times 50 \right) \text{J kg K}^{-1} \times \frac{3}{8.64 \times 10^4 \times 365 \times 100} \text{ K s}^{-1} \\ &= 0.15 \text{ W m}^{-2} \end{aligned}$$

A flux of this magnitude is more than an order of magnitude or more below the "noise level" inherent in the measurements and would therebore be undetectable.

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10.8 On the basis of the global energy balance in Fig. 10.2, estimate what the surface temperature of the earth would be in the absence of latent and sensible heat fluxes. (For purposes of this problem, assume that the fraction of the longwave radiation emitted from the earth's surface that is absorbed by the atmosphere and reemitted back to the earth's surface is unchanged.)

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Solution: In the current global energy balance

$$F_s = F_s = F_{LW}^{\uparrow} - F_{LW}^{\downarrow} + F_h + F_m \tag{1}$$

where F_s is the net solar radiation absorbed at the earth's surface, $F_{LW}^{\uparrow}$ is the upward emission of longwave radiation from the earth's surface, $F_{LW}^{\downarrow}$ is the downward emission of longwave radiation from the atmosphere, F_h is the sensible heat flux and F_m is the latent heat flux. In the absence of sensible and latent heat fluxes the balance would be

$$F_s = \mathbf{F}_{LW}^{\uparrow} - \mathbf{F}_{LW}^{\downarrow} \tag{2}$$

where the quantities in boldface type refer to conditions that would prevail in the absence of latent and sensible heat fluxes. If we assume that

$$\frac{\mathbf{F}_{LW}^{\downarrow}}{\mathbf{F}_{LW}^{\uparrow}} = \frac{F_{LW}^{\downarrow}}{F_{LW}^{\uparrow}}$$

we can write

$$\mathbf{F}_{LW}^{\uparrow} = F_s \left(1 - \frac{F_{LW}^{\downarrow}}{F_{LW}^{\uparrow}}\right)^{-1}$$

Substituting from the global energy balance figure

$$\mathbf{F}_{LW}^{\uparrow} = 168 \left(1 - \frac{324}{390}\right)^{-1} = 988 \text{ W m}^{-2}$$

compared to $F_{LW}^{\uparrow} = 390$ W m^{-2}. From the Stefan-Boltzmann law it follows that

$$\mathbf{T} = T \left(\frac{\mathbf{F}_{LW}^{\uparrow}}{F_{LW}^{\uparrow}}\right)^{1/4} = 288 \left(\frac{\mathbf{F}_{LW}^{\uparrow}}{390}\right)^{1/4} = 363 \text{ K}$$

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10.9 On the basis of the data in Fig. 10.3 estimate the poleward flux of energy across 40°N by the atmosphere and oceans. Hint: Construct a rectangle with a base extending from 10-90°N and a height such that the area of the rectangle is equal to the area under the curve in this latitude belt. This height is the area average deficit in net radiation poleward of 40°N. The energy flux across 40°N must be equal to this deficit times the area

$$2\pi \int_{40}^{90} \cos\phi d\phi$$

where R_E is the radius of the earth.

-

10.10 Compare the daily insolation upon the top of the atmosphere (a) at the North Pole at the time of the summer solstice, and (b) at the equator at the time of the equinox . The solar declination angle at the time of the summer solstice is 23.45° and the the Earth-Sun distances are 1.52 and 1.50 × 10^8 km, respectively. *Answers* (a) 46.4 versus 38.0 MJ m^{-2} day^{-1}(see also Fig. 10.11).

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Solution: (a) At the time of the solstice over the North Pole the solar zenith angle is the complement of the declination angle and is independent of time of day. Hence

$$Insolation = 1366 \text{ W m}^{-2} \times \left(\frac{1.50}{1.52}\right)^2 \cos\left(90^\circ - 23.5^\circ\right) \times 8.64 \times 10^4 \text{ s day}^{-1} = 45.8 \ \times 10^6 \text{ J m}^{-2} \text{ per day}$$

(b) At the time of the equinox the sun is overhead on the equator . The daily averaged flux may be obtained by calculating the solar radiation intercepted by the earth per unit width of latitude along the equator (1366 W m^{-2} × $2R_E$) and dividing by the circumference of the earth $2\pi R_E$. Hence,

$$Insolation = \frac{1366}{\pi} \times 8.64 \times 10^4 \text{ s day}^{-1} = 37.6 \ \times 10^6 \text{ J m}^{-2} \text{ per day}$$

Alternatively, the answer can be obtained by integrating the flux over the 12 daylight hours, expressing the solar zenith angle as

$$\phi = 2\pi\left(t - t_0\right)/24 \text{ h}$$

where t_0 is local noon.

$$Insolation = 1366 \text{ W m}^{-2} \times \frac{8.64 \times 10^4 \text{ s}}{2\pi} \int_{-\pi/2}^{\pi/2} \cos\phi d\phi$$

where the numerical value of the integral is 2.

-

10.11 As cold, continental air passes over the Gulf Stream on a certain winter day, the temperature rises by 10 C over a distance of 300 km. Within this interval the mean mixed layer depth is 1 km and the average wind speed is 10 m s^{-1}. No condensation is taking place within the lowest km and the radiative fluxes are negligible. Calculate the sensible heat flux from the sea surface.

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Solution: The sensible heat flux is

$$F = \frac{mc_p \triangle T}{\triangle t}$$

where m is the mass per unit area of the air in the mixed layer, c_p is the specific heat, $\triangle T$ is the temperature rise and $\triangle t$ is the time interval over which it takes place. The mass per unit area of the air in the lowest kilometer of the atmosphere can be estimated only roughly. (I should have prescribed that the layer be 100-hPa thick.) Depending on how you estimate it you should come up with a value of 110 hPa plus or minus 10%. With this value

$$m = \frac{110 \times 10^2 \text{ hPa}}{9.8 \text{ m s}^{-2}} = 1120 \text{ kg m}^{-2}$$

The time required for the air to traverse the 300 km distance is

$$\frac{300 \times 10^3 \text{ m}}{10 \text{ m s}^{-1}} = 3 \times 10^4 \text{ s}$$

Hence

$$F = \frac{1.120 \times 10^3 \text{ kg m }^{-2} \times 1004 \text{ J kg}^{-1} \text{ K}^{-1} \times 10 \text{ K}}{3 \times 10^4 \text{ s}} = 375 \text{ W m}^{-2}$$

10.12 In the absence of fluxes at the earth's surface, estimate rougly how long it would take the effective temperature of the atmosphere to drop by a factor of e if the sun were to abruptly stop emitting energy. How would you expect your answer to compare (qualitatively) to the corresponding *radiative relaxation times* for the atmospheres of Mars and Venus, or for Earth if fluxes at the air-sea interface were taken into account.

-

Solution:

$$-\frac{c_p p_0}{g}\frac{dT_E}{dt} = \sigma T_E^4$$

Rearranging yields

$$-\frac{dT_E}{T_E^4} = \frac{\sigma g}{c_p p_0} dt$$

Integrating, we obtain

$$\begin{aligned} t &= -\frac{c_p p_0}{\sigma g}\int_{T_E/e}^{T_E} \frac{dT_E}{T_E^4} \\ &= \frac{c_p p_0}{3\sigma g}\left(\frac{1}{(T_E/e)^3} - \frac{1}{T_E^3}\right) \\ &= \frac{c_p p_0}{3\sigma g T_E^3}\left(e^3 - 1\right) \end{aligned}$$

Substituting numerical values we obtain

$$\begin{aligned} t &= \frac{1004 \times 10^5}{3 \times 5.67 \times 10^{-8} \times 9.8 \times (255)^3}(20.08 - 1) \\ &= 7.2 \times 10^7 \text{ s} \\ &= 833 \text{ days} \end{aligned}$$

10.13 Prove that x, the response of a system to the forcing F the presence of feedbacks with a total gain of g is given by

$$\frac{dx}{dF} = \frac{\partial x}{\partial F}\left(1 + g + g^2 + ...g^n + ...\right)$$

Give a physical interpretation of this result.

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10.14 Consider a system with two positive two feedback processes capable of amplifying the response to a given forcing by factors of 2.5 and 2.0, respectively, if each were acting in isolation. How much would they amplify the forcing of the system if they were acting in combination.

Solution: The sensitivity of the response x to the forcing F is

$$\frac{dx}{dF} = \frac{\partial x/\partial F}{1-g}$$

where g is the gain due to the presence of the feedback process(es). Solving for g we obtain

$$g = \frac{dx/dF - \partial x/\partial F}{dx/dF}$$

Hence the gain for the first feedback process is

$$g_1 = \frac{2-1}{2} = 0.5$$

and the gain for the second is

$$g_2 = \frac{2.5-1}{2.5} = 0.6$$

The combined gain $(g_1 + g_2)$ is greater than 1.0. Hence, the system is unstable with respect to the forcing in the presence of the combined feedbacks.

10.15 The thermal expansion coefficient for water is xxx, mass of water in the oceans is equivalent to xxx kg per square meter of the earth's surface and the fraction of the area of the earth covered by oceans is 70%. Estimate the amount by which would sea-level rise