

1.5 Prove that exactly half the area of the earth lies equatorward of 30° latitude.

$$\begin{aligned}
 \text{Area} &= \int_0^{2\pi} \int_{-30^\circ}^{30^\circ} R_E^2 \cos \phi d\phi d\lambda \\
 &= 2\pi R_E^2 \int_{-30^\circ}^{30^\circ} \cos \phi d\phi \\
 &= 2\pi R_E^2 \sin \phi \Big|_{-30^\circ}^{30^\circ} \\
 &= 2\pi R_E^2 (0.5 - (-0.5)) \\
 &= 2\pi R_E^2, \text{ half the area of a sphere.}
 \end{aligned}$$

1.6 How many days would it take a hot air balloon traveling eastward along 40°N at a mean speed of 15 m s^{-1} to circumnavigate the globe?

The length of the latitude circle is $l = 2\pi R_E \cos 40^\circ = 2\pi \times 6.37 \times 10^6 \times 0.766 = 30.7 \times 10^6 \text{ m}$ or 30,700 km.

The time is the distance divided by the velocity, i.e.,

$$t = \frac{l}{v} = \frac{30.7 \times 10^6 \text{ m}}{15 \text{ m s}^{-1}} = 2.05 \times 10^6 \text{ s} = 23.7 \text{ days}$$

1.7 Prove that pressure expressed in cgs units of millibars ($\text{mb} = 10^{-3}\text{b}$) is numerically equal to pressure expressed in SI units of hPa (hectoPascals = 10^2 Pa).

$$\begin{aligned}
 1 \text{ mb} &= 10^3 \frac{\text{dynes}}{\text{cm}^{-2}} = 10^3 \frac{\text{g cm}}{\text{s}^{-2} \text{cm}^{-2}} = 10^3 \frac{\text{g}}{\text{s}^{-2} \text{cm}} \\
 &= 10^3 \frac{10^{-3} \text{ kg}}{\text{s}^{-2} 10^{-2} \text{ m}} = \frac{10^2 \text{ kg m}}{\text{s}^{-2} \text{m}^2} = 10^2 \frac{\text{N}}{\text{m}^{-2}} = 1 \text{ hPa}
 \end{aligned}$$

1.8 How far below the surface of the water does a diver experience a pressure of 2 atmospheres (i.e., a doubling of the ambient atmospheric pressure due to the weight of the overlying water). Answer $\sim 10 \text{ m}$.

At the depth where the pressure is 2 atm, half the pressure is due to the weight of the overlying air and the other half (1 atm) is due to the weight of the overlying water, which is equal to the density times the depth times g . The density of water is $\rho = 1000 \text{ kg/m}^3$.

$$\begin{aligned}
 p &= \rho g z = 10^5 \text{ Pa}; \\
 z &= 10^5 / (10^3 \times 9.8) \simeq 10 \text{ m}
 \end{aligned}$$

1.9 In a sounding taken on a typical winter day at the South Pole the temperature at the ground is -80°C and the temperature at the top of a 30 m high tower is -50°C . Estimate the lapse rate within the lowest 30 m, expressed in K km^{-1} .

$$\Gamma \equiv -\frac{\Delta T}{\Delta z} = -\frac{(-50 - (-80))}{30} = -1^\circ\text{C m}^{-1} = -1,000^\circ\text{C km}^{-1}, \text{ a very strong inversion!!}$$

1.10 "Cabin altitude" in typical commercial airliners is around 1.7 km. Estimate the typical pressure and density of the air in the passenger cabin.

Let us assume values at sea-level of $p = 1,000 \text{ hPa}$ and $\rho = 1.25 \text{ kg m}^{-3}$ and exponential relationships of the form

$$p = p_0 e^{-z/H}$$

and

$$\rho = \rho_0 e^{-z/H}$$

where the scale height $H = 7.5 \text{ km}$. Substituting $z = 1.7 \text{ km}$, we obtain $p \sim 800 \text{ hPa}$ and $\rho \sim 1.00 \text{ kg m}^{-3}$.

1.11 Prove that density and pressure, which decrease more or less exponentially with height, decrease by a factor of 10 over a depth of $\ln 10 = 2.303$ times the scale height.

If $p = p_0 e^{-z/H}$, then $p/p_0 = 0.1$ at the level where $e^{-z/H} = 0.1$ or $z = -H \ln(0.1)$ or $z = H \ln(10) = 2.303H$.

1.12 Consider a perfectly elastic ball of mass m bouncing up and down on a horizontal surface under the action on a downward gravitational acceleration g . Prove that in the time average over an integral number of bounces, the downward force exerted by the ball upon the surface is equal to the weight of the ball. [**Hint:** The downward force is equal to the the downward momentum imparted to the surface with each bounce divided by the time interval between successive bounces.] Does this result suggest anything about the "weight" of an atmosphere comprised of gas molecules?

The momentum imparted to the surface with each bounce is $\Delta p = 2mv_0$. The time required for the ball to fall from the top of its orbit to the surface is $t_1 = v_0/g$, so the time between bounces must be $\Delta t = 2v_0/g$. The average

downward force is equal to the average rate of transfer of momentum $F = \frac{\Delta p}{\Delta t} = 2mv_0 / (2v_0/g) = mg$, which is the weight of the ball.

1.13 Estimate the percentage of the mass of the atmosphere that resides in the stratosphere based on the following information. The mean pressure level of tropical tropopause is around 100 hPa and that of the extratropical tropopause is near 300 hPa, where the break between the tropical and extratropical tropopause occurs near 30° latitude, in which case, exactly half the area of the earth lies in the tropics and half in the extratropics. On the basis of an inspection of Fig. 1.7, verify that these estimates are reasonably close to observed conditions.

The boundary between the tropical tropopause and the extratropical tropopause is close to 30° latitude. Roughly half the mass of the atmosphere lies equatorward of this latitude. In the tropical half of the globe the tropopause level is close to 100-hPa. At any height, pressure is the weight of the atmosphere above it (per unit area), so we can estimate the ratio r of stratospheric to total mass as $r = \frac{m_{strat}}{m_{tot}} = \frac{p_{strat}}{p_0}$, where p_0 is the surface pressure (the mass above the stratopause is negligible). In the tropics $r = \frac{100}{1000} \sim 0.1$. Roughly 10% of the mass of the atmosphere lies in the stratosphere. In the extratropical half of the globe the tropopause lies close to 300-hPa, so roughly 30% of the mass of the atmosphere lies in the stratosphere. So for the globe as a whole the fraction of the mass of the atmosphere that lies in the stratosphere is the mean of these two values, or $\sim 20\%$.

1.14 If the earth's atmosphere consisted of an incompressible fluid whose density was everywhere equal to that observed at sea level (1.25 kg m^{-3}) how deep would it have to be to account for the observed mean surface pressure of $\sim 10^5 \text{ Pa}$.

If h is the height of the free surface of the fluid

$$p = \rho gh = 1000 \text{ hPa}$$

Solving, we obtain

$$h = \frac{10^5}{9.8 \times 1.25} \simeq 8,000 \text{ m}$$

1.15 If the density of air decreases exponentially with height from a value of 1.25 kg m^{-3} at sea-level, calculate the scale height that is consistent with the observed global mean surface pressure of $\sim 10^5 \text{ hPa}$. [**Hint:** Integrate the counterpart of (1.4) from the earth's surface to infinity to obtain the atmospheric mass per unit area.]

Integrating (1.4) from sea-level to the top of the atmosphere assuming a constant value of the scale height yields

$$p_0 = \int_0^\infty \rho g dz = \int_0^\infty \rho_0 e^{-z/H} g dz = \rho_0 g \int_0^\infty e^{-z/H} dz = \rho_0 g H e^{-z/H} \Big|_0^\infty = \rho_0 g H$$

Solving, we obtain $H = p_0/\rho_0g = 10^5/(1.25 \times 9.8) \sim 8,000$ m

1.16 The equatorward flow in the tradewinds is on the order of 1 m s^{-1} averaged around the circumference of the earth at 15°N and 15°S , and it extends through a layer extending from sea-level up to around the 850 hPa pressure surface. Estimate the equatorward mass flux into the equatorial zone due to this circulation. **Hint:** The equatorward mass flux across the 15°N , in units of kg s^{-1} is given by

$$-\oint_{15^\circ\text{N}} \int_0^{z_{850}} \rho v dx dz$$

where ρ is the density of the air, v is the meridional (northward) velocity component, the line integral denotes an integration around the 15°N latitude circle and the vertical integral is from sea-level up to the height of the 850 hPa surface. Evaluating the integral, make use of the relations

$$\oint_{15^\circ\text{N}} dx = 2\pi R_E \cos 15^\circ (= l, \text{ length of the latitude circle})$$

and

$$\int_0^{z_{850}} \rho dz = -\partial p/g = \frac{p_0 - p_{850}}{g} (= m, \text{ mass per unit area in the layer})$$

The mass flux, per unit length along the latitude circle is mv . Multiplying by the length of the latitude circle and by 2 (the flux enters the equatorial region from 15°S and 15°N) we obtain the total mass flux

$$\begin{aligned} F_m &= 2 \times l \times m \times v = \\ &= 2 \times 2 \times \pi \times R_E \times \cos(15) \times 1.5 \times 10^4 / 9.8 \times 1 \\ &= 11.8 \times 10^{10} \text{kg s}^{-1}. \end{aligned}$$

1.17 During September, October and November the mean surface pressure over the northern hemisphere increases at a rate of ~ 1 hPa per 30-day month. Calculate the mass averaged northward velocity across the equator that is required to account for this pressure rise. **Hint:** Assume that atmospheric mass is conserved: i.e, that the pressure rise in the northern hemisphere is entirely due to the influx of air from the southern hemisphere. By defining the mass averaged northward velocity

$$v_m \equiv \frac{\oint \int_0^\infty \rho v dx dz}{\oint \int_0^\infty \rho dx dz}$$

we can write the mass flux in the form

$$F_m = v_m \oint \int_0^\infty \rho dx dz$$

which can be solved for v_m .]

The required mass transport F_m is equal to the rate of pressure change, divided by g , times the area of the hemisphere ($2\pi R_E^2$), or

$$F_m = \frac{1}{g} \times \frac{\delta p}{\delta t} \times 2\pi R_E^2 = v_m \oint \int_0^\infty \rho dx dz$$

where δt is 1 month or 2.59×10^6 s. Noting that $\oint dx = 2\pi R_E$, and $\int_0^\infty \rho dz = p_0/g$, we can write

$$\frac{\delta p}{g \times \delta t} \times 2\pi R_E^2 = 2\pi R_E \times v_m \times p_0/g$$

Solving, we obtain .

$$v_m = \frac{\delta p \times R_E}{p_0 \delta t} = 100 \times \frac{6.37 \times 10^6}{10^3 \times 2.59 \times 10^6} = 2.46 \text{ mm s}^{-1}$$